Adaptive Optics Reconstruction Methods

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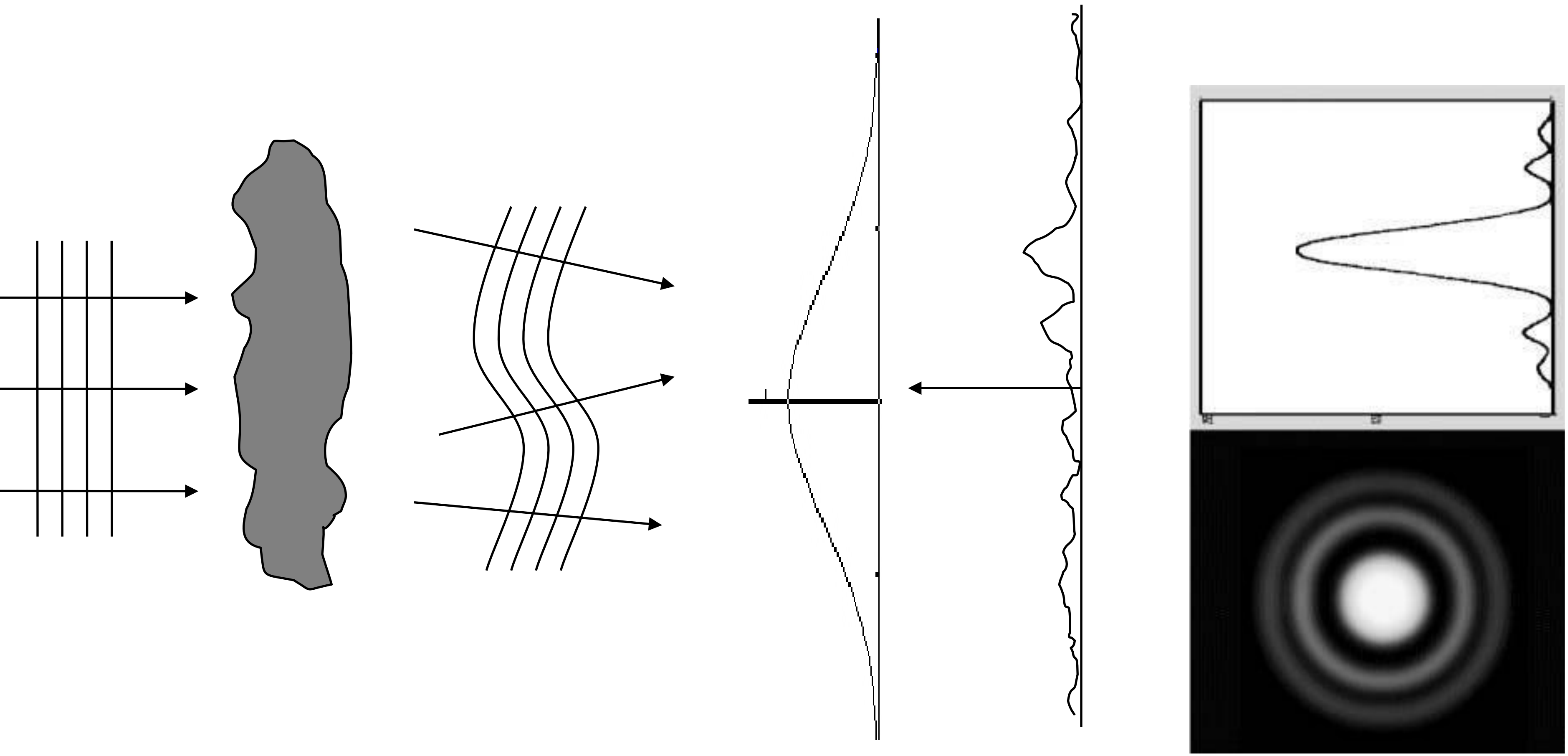
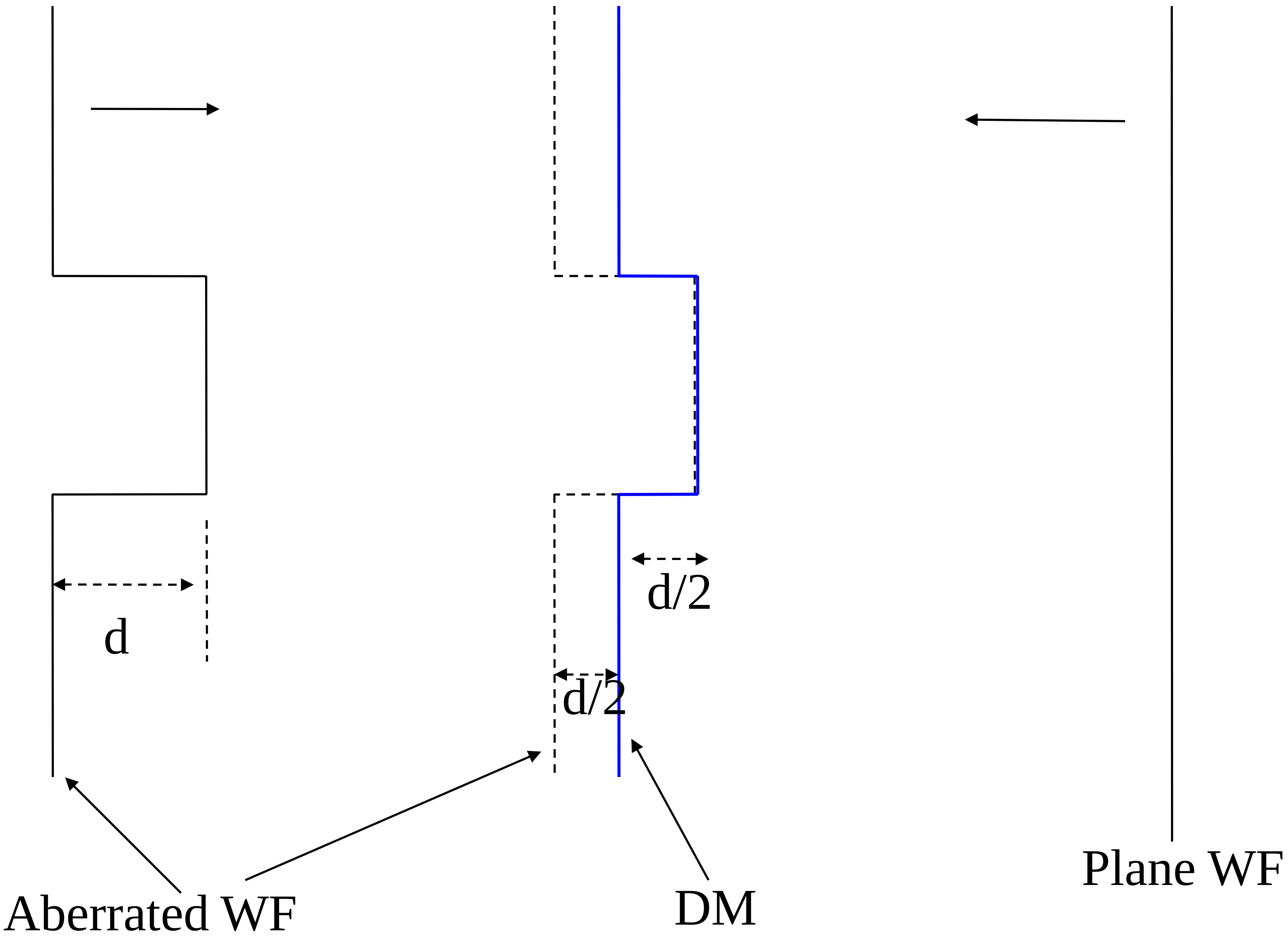
Why do we need plane wavefront ?



a b c

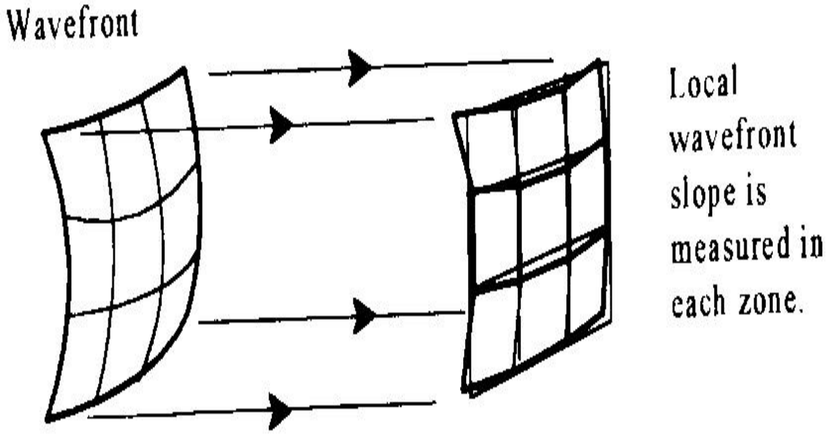


d
d/2
d/2
Aberrated WF
DM
Plane WF

Wave Front Sensing

- Zonal wavefront sensing

  Wavefront is expressed in terms of OPD over a small spatial area (zone/sub-aperture)



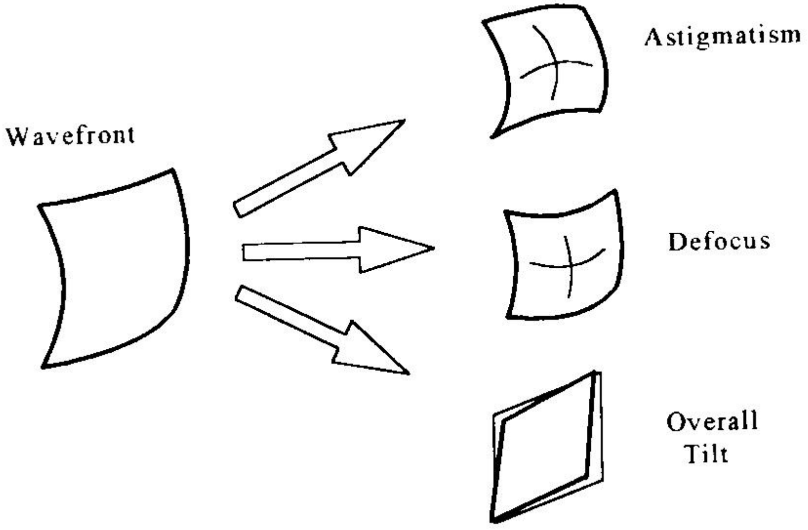
(a) Zonal wavefront sensing.

Wave Front Sensing

- Modal Wavefront Sensing

  Wavefront is expressed in terms of some polynomial expansion over the entire pupil

  (i.e. PM)



(b) Modal wavefront sensing.

Zernike Polynomials

- WF can be expressed as Zernike polynomials
- Orthogonality of Zernike polynomials over unit circle
- Annular Zernike Polynomials
- Examples:

$$z_1 = 1 \qquad piston$$

$$z_2 = 2r\cos\theta \qquad x\ \ tilt$$

$$z_3 = 2r\sin\theta \qquad y\ \ tilt$$

$$z_4 = \sqrt{3}(2r^2 - 1) \qquad defocus$$

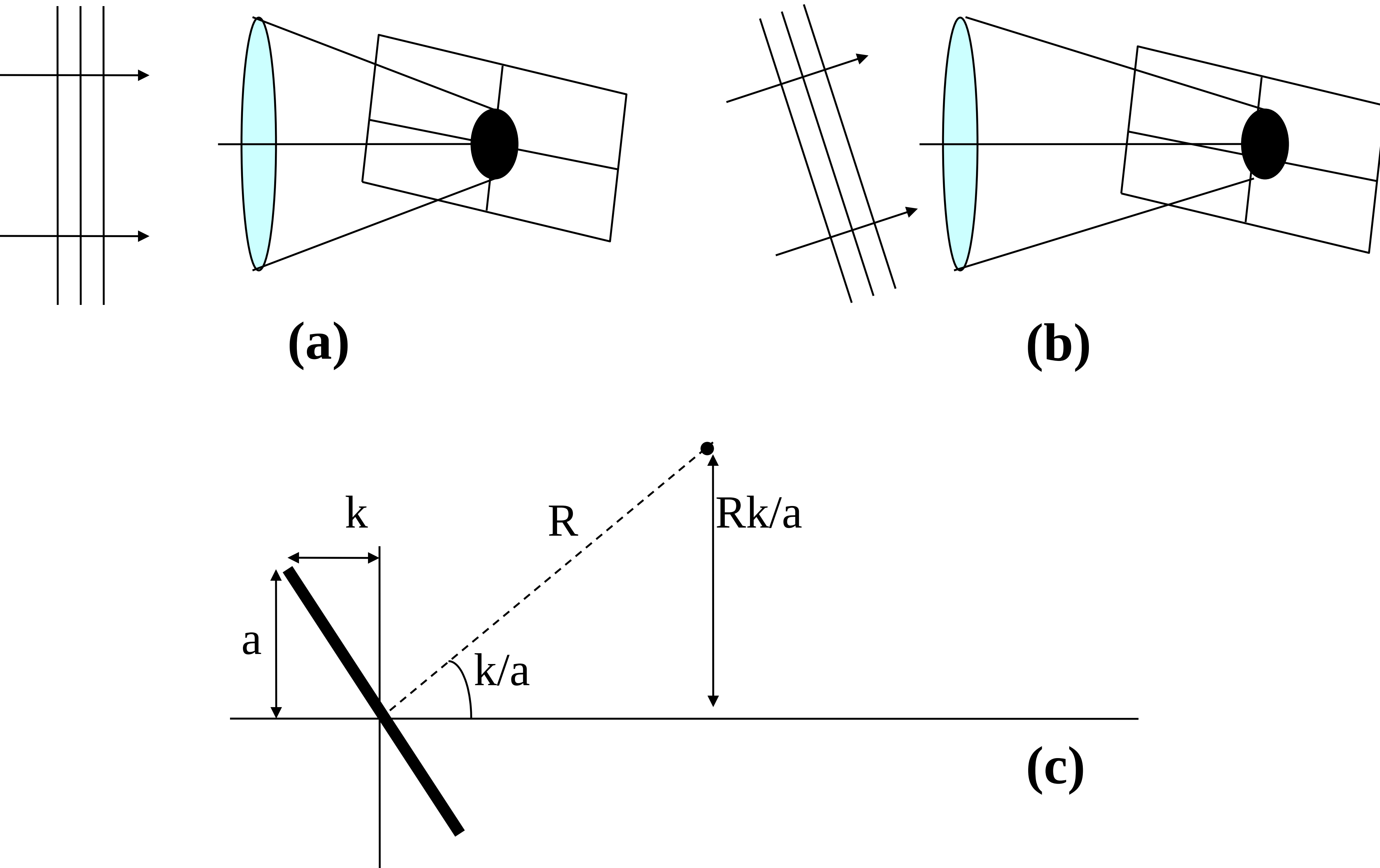
$$z_5 = \sqrt{6}r^2\sin 2\theta \qquad astigmatism$$

$$z_6 = \sqrt{6}r^2\cos 2\theta \qquad astigmatism$$

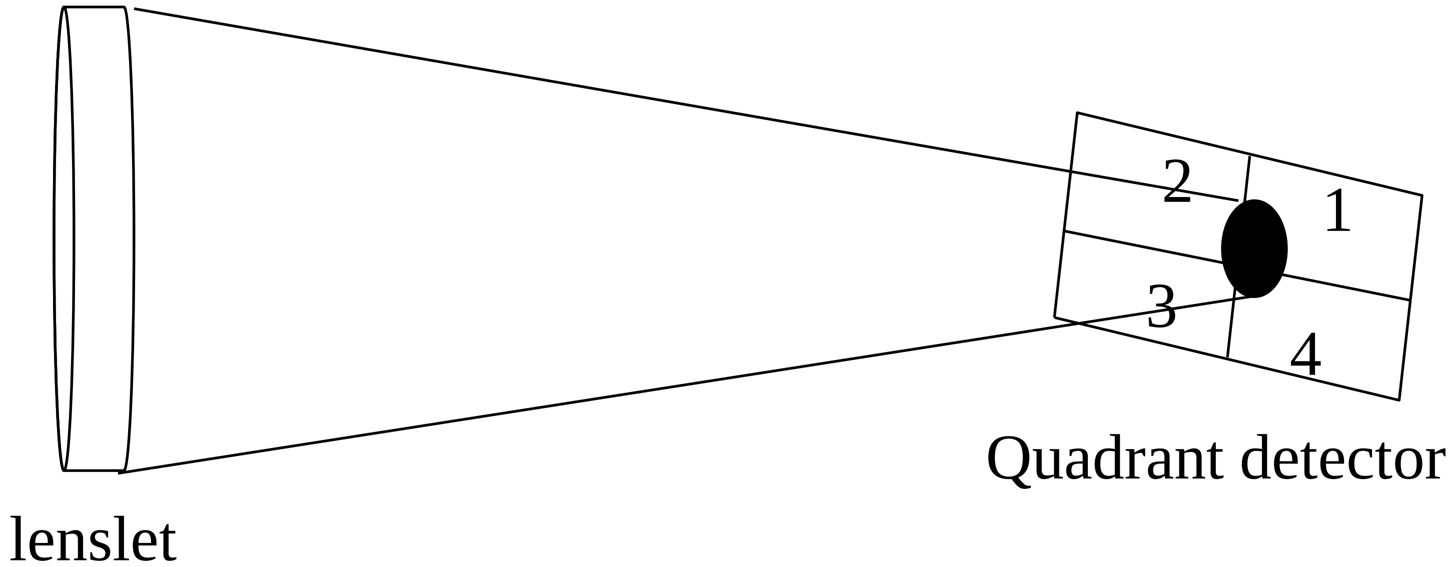
Zonal Wavefront Sensing

- Sub-aperture size & Fried parameter: Refractive Index does not change over this region for the time interval of the order of , for example, 10 ms.
- Local tilts are measured.
- Centroid of intensity distribution is equal to shifted origin of the image plane; shift being proportional to the tilt (fig. (c) in next slide) , i.e. local tilt shifts centroid.

Zonal sensing



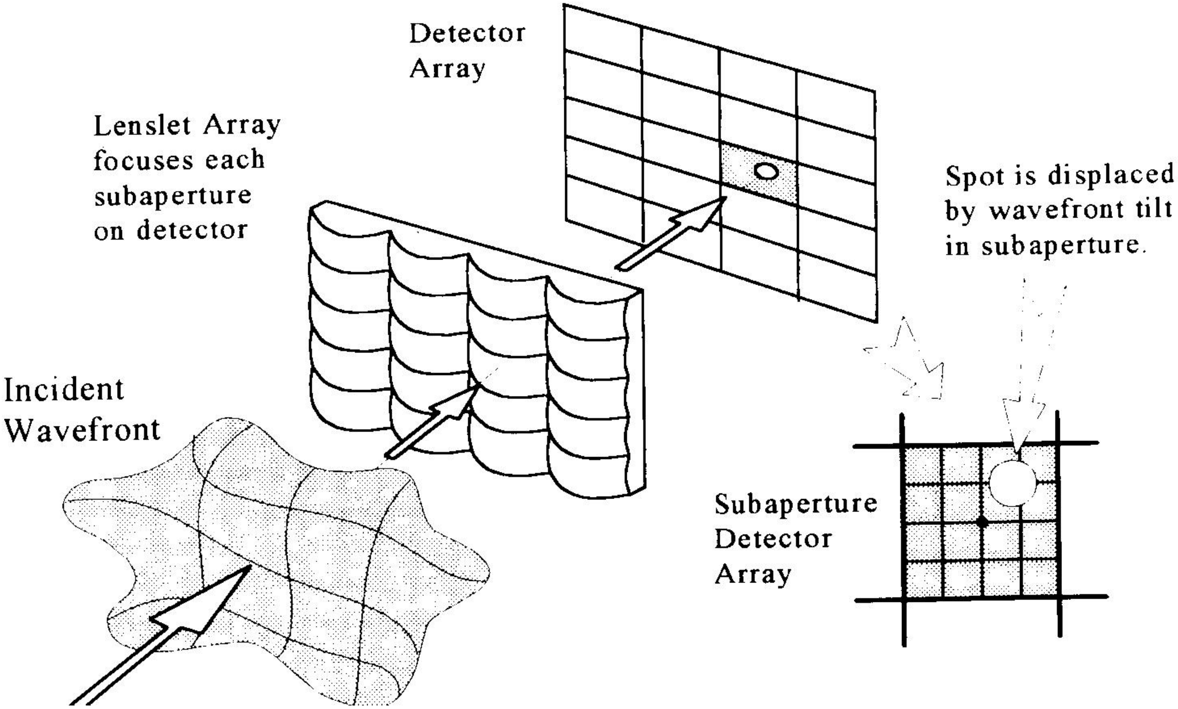
Zonal sensing



$$x_{centroid} = \frac{\int_1 Ids + \int_4 Ids - \int_2 Ids - \int_3 Ids}{\sum_i \int_i Ids}$$

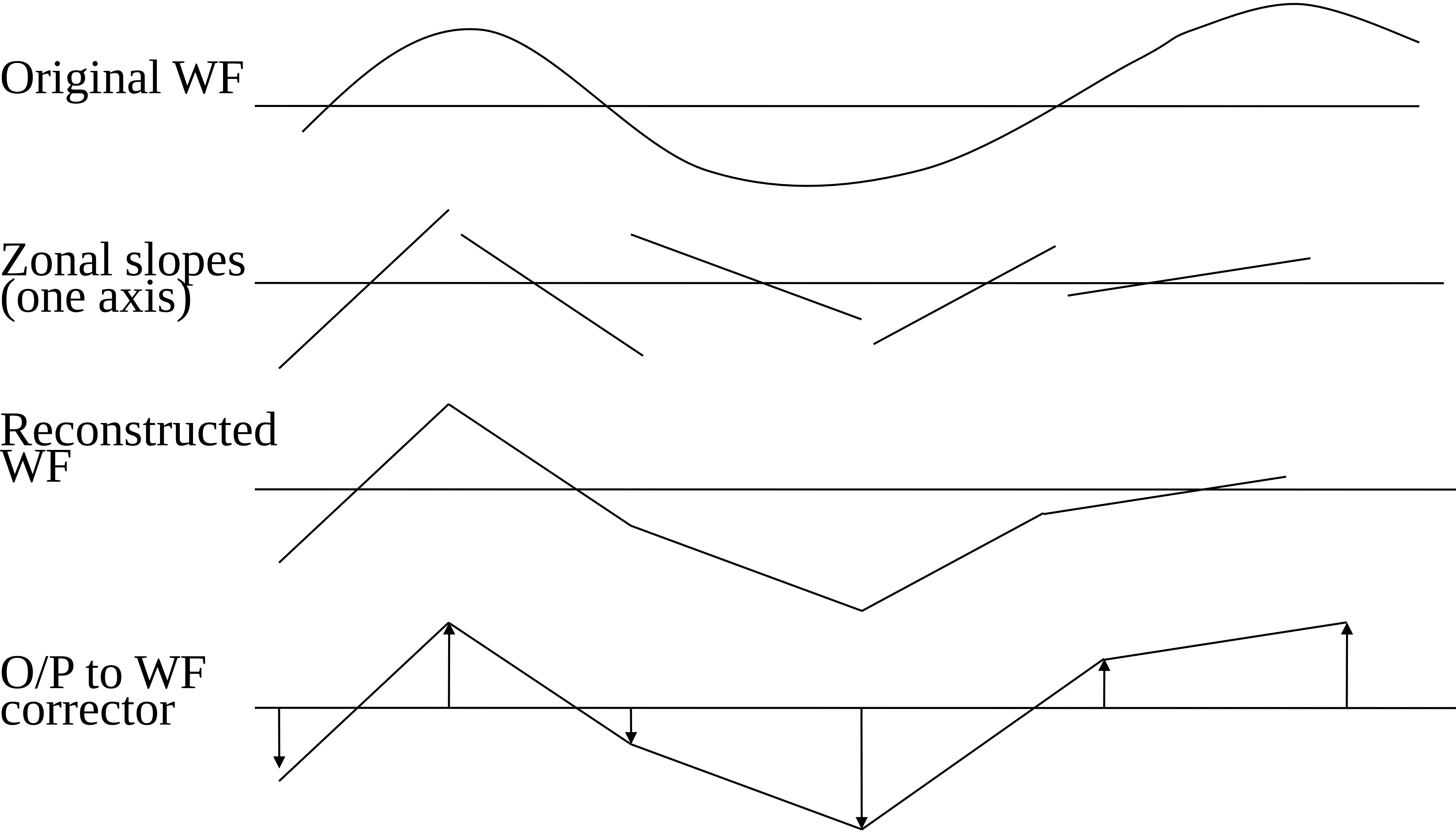
$$y_{centroid} = \frac{\int_1 Ids + \int_2 Ids - \int_4 Ids - \int_3 Ids}{\sum_i \int_i Ids}$$

Shack-Hartmann Sensor



$$s(x,y) = \alpha(x,y) \cdot z$$

WF surface from slope measurements



Wavefront Reconstruction

(Phase/OPD from WF slopes)

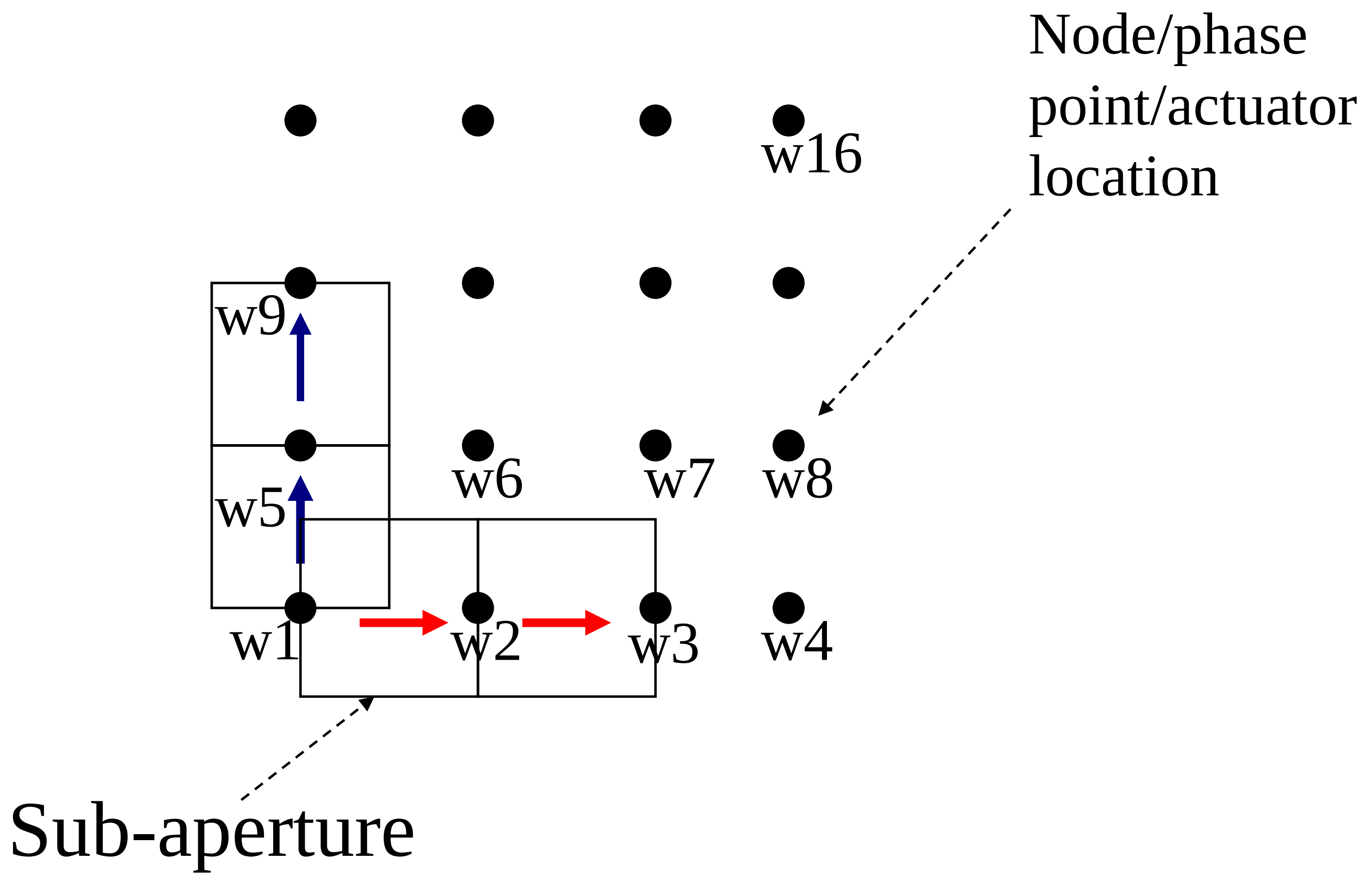
- OPD at various points on WF is determined from WF slopes at other points on the WF
- Number of WF slope measurements (M) is generally > Number of unknown phase points (N)
- Problem of phase determination from WF slopes depends on geometry of arrangement of actuators w.r.t. phase points on WF

Wavefront Reconstruction

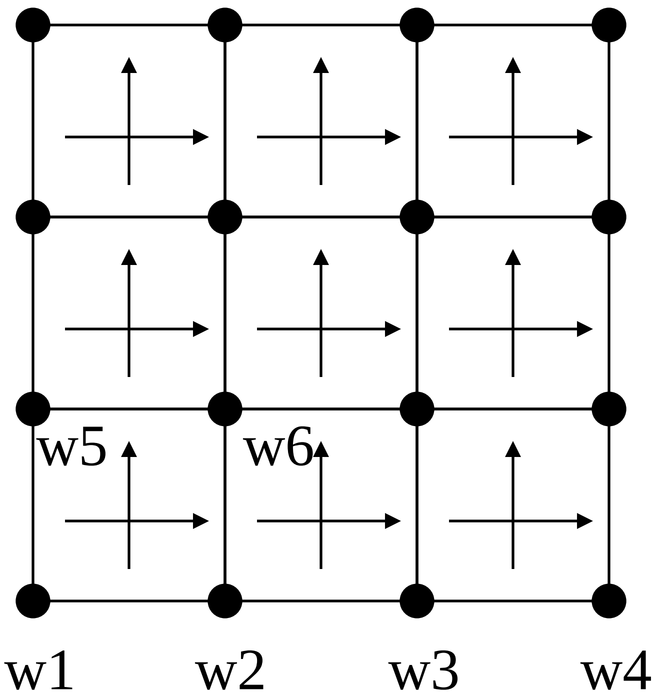
Main 3 Geometries :

- Hudgin
- Fried
- Southwell

Hudgin Geometry



Fried Geometry

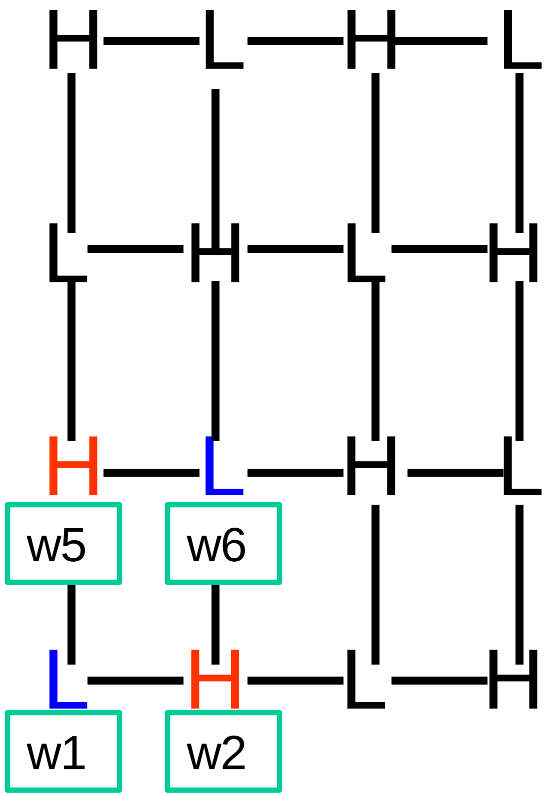


Slope Model

$$\left(\frac{w_2 + w_6}{2}\right) - \left(\frac{w_1 + w_5}{2}\right) = s^x$$

$$\left(\frac{w_5 + w_6}{2}\right) - \left(\frac{w_1 + w_2}{2}\right) = s^y$$

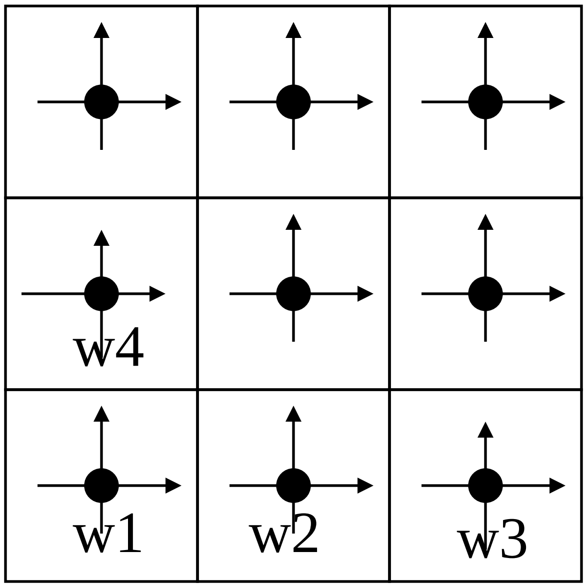
Waffle Mode (Fried geometry)



- Two actuators on one diagonal are equally high and two actuators on the other diagonal are equally low.
- Centroid is undisturbed relative to centroid of flat wavefront.
- Removal of waffle mode:

$$\frac{w_2 + w_5}{2} - \frac{w_1 + w_6}{2} = 0$$

Southwell Geometry



Slope Model

$$\frac{s_1^x + s_2^x}{2} = w_2 - w_1$$

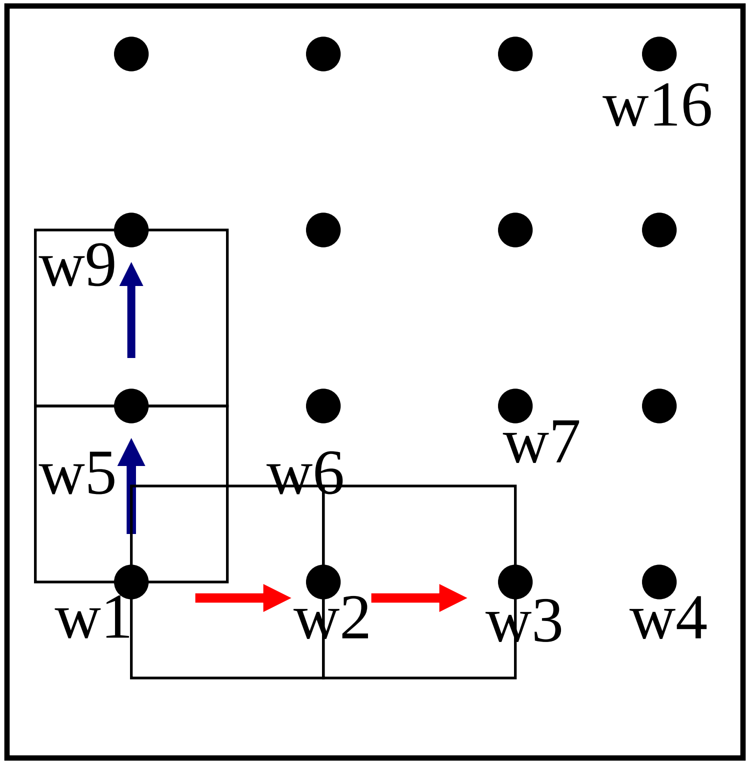
$$\frac{s_1^y + s_4^y}{2} = w_4 - w_1$$

Slope Model (Hudgin Geometry)

$$w_2 - w_1 = s_1^x \qquad w_5 - w_1 = s_1^y$$

$$w_3 - w_2 = s_2^x \qquad w_6 - w_2 = s_2^y$$

$$\vdots \qquad\qquad \vdots$$



$$
\begin{bmatrix} s_1^x \\ s_2^x \\ \cdot \\ \cdot \\ s_M^x \\ s_1^y \\ s_2^y \\ \cdot \\ \cdot \\ s_M^y \\ 0 \end{bmatrix}
=
\begin{bmatrix}
-1 & 1 & & & & & & \\
 & -1 & 1 & & & & & \\
 & & & & \cdot & & & \\
 & & & & \cdot & & & \\
 & & & & \cdot & & & \\
-1 & & & & 1 & & & \\
 & -1 & & & & 1 & & \\
 & & & & & \cdot & & \\
 & & & & & \cdot & & \\
 & & & & & \cdot & & \\
1 & 1 & \cdot & \cdot & \cdot & & 1 & 1
\end{bmatrix}
\begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ w_N \end{bmatrix}
$$

$$S_{M\times 1} = A_{M\times N} \cdot W_{N\times 1}$$

Slope measurements from SH sensor

From geometry

To be estimated

Least Squares Method

$$\rightarrow [A]\cdot[W] = [S]$$

$$\rightarrow \text{Residual vector}[r] = [A]\cdot[W] - [S]$$

$$\rightarrow \|r\| = \left\|A\overline{W} - S\right\| \leq \|AW - S\| \quad for\, all \;\; W$$

$$\rightarrow \overline{W} = (A^T A)^{-1} A^T S$$

Difficulty: Every time $A^T A$ cannot be inverted.

Least squares method

$$\chi^2 = \sum_{i=1}^{M} \left[S_i - \sum_{k=1}^{N} A_{ik} \cdot W_i \right]^2$$

$$\frac{\partial}{\partial w_i} \left(\chi^2 \right) = 0$$

$$W = \left[A^T A \right]^{-1} \cdot \left[A^T \right] \cdot S$$

Use of SVD

- Every MxN matrix can be factorized into 3 matrices

$$[A] = [U][D][V^T]$$

where

$$[D] = \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & d_n \end{bmatrix}$$

$$[A^+] = [V][D^{-1}][U^T]$$

Psuedo-inverse:

$$[A^+] = [V] \begin{bmatrix} \frac{1}{d_1} & & & & \\ & \frac{1}{d_2} & & & \\ & & . & & \\ & & & . & \\ & & & & \frac{1}{d_N} \end{bmatrix} [U^T]$$

$$[D^{-1}] = \frac{1}{d_i} \qquad if \;\; d_i > t$$

$$= 0 \qquad otherwise$$

$$where \;\; t \;\; is \;\; a \;\; small \;\; threshold.$$

Finally we get the solution as:

$$[\overline{W}] = [A^{+}][S]$$

$$= [V][D^{-1}][U^{T}][S]$$

References

- Adaptive Optics for Astronomical Telescopes; John W. Hardy; OUP 1998

- Linear Algebra; Gilbert Strang; Thomson/Brooks,Cole

- Wave-front Reconstruction from wave-front slope measurements; W.H. Southwell; J.Opt.Soc.Am.,Vol.70, August 1980