Solve the problems listed below, and write up your answers clearly and completely. Do not turn in rough work – instead, make a clean copy after checking your calculations. Use English sentences and phrases to explain your solution and describe key equations. Show your work!

1. A “toy model” for a gas giant planet, as discussed in class, is defined by the hydrostatic equilibrium equation,
\[
\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho(r),
\]
the mass continuity equation,
\[
\frac{dM}{dr} = 4\pi r^2\rho(r),
\]
and a simple “polytropic” equation of state,
\[
P = K\rho^2.
\]
Here \(K\) is a constant with units of \(m^5\) kg\(^{-1}\) s\(^{-2}\). The solution to these model equations is the density profile
\[
\rho(r) = \rho_c \sin(r \pi/R) r \pi/R,
\]
where \(\rho_c\) is the mass density at the center and \(R\) is the outer radius of the planet.
(a) Use equations (2) and (4), along with the boundary condition \(M(0) = 0\) to find \(M(r)\), the total mass enclosed within radius \(r \leq R\).
(b) Check your result for part (a) by verifying that the average density within the planet’s radius \(R\) is \(\bar{\rho} = 3\rho_c/\pi^2\).
(c) Use equations (3) and (4) to express the pressure \(P\) as a function of \(r\). Differentiate \(P\) with respect to \(r\), compare the result to the right-hand-side of equation (1), and show that the two are equal if \(R = \sqrt{\pi K/2G}\).
(d) Jupiter’s mass is \(M_J \approx 1.90 \times 10^{27}\) kg, and its radius is \(R_J \approx 7.0 \times 10^7\) m. What are the corresponding values for \(K\) and \(\rho_c\)? (Note: these values “fit” the model to Jupiter.)
(e) Using the values of \(K\) and \(\rho_c\) which fit Jupiter, what central pressure \(P_c\) does this model predict? Compare the result to the central pressure \(P_c = \frac{3GM_J^2}{8\pi R_J^4}\) predicted for a homogeneous (constant density) model of Jupiter. Which \(P_c\) is greater, and why?

2. Saturn radiates \(L_{\text{rad}} \approx 1.98 \times 10^{17}\) kg m\(^2\) s\(^{-3}\) of infrared radiation, even though it absorbs only \(W_{\text{sol}} \approx 1.11 \times 10^{17}\) kg m\(^2\) s\(^{-3}\) of solar radiation. Suppose that the difference is entirely provided by gravitational energy released by gradual contraction. Saturn’s mass is \(M_S \simeq\)
5.7 \times 10^{26} \text{ kg}, and its current radius is \( R_S \simeq 5.8 \times 10^7 \text{ m}. \) To calculate the gravitational energy, you may assume that Saturn has uniform density, and use the formula appropriate for a homogeneous planet.

(a) How fast is Saturn’s radius changing; that is, what is \( dR_S/dt? \)

(b) Assuming it continued contracting at this rate for \( 10^9 \text{ yr}, \) by what percentage would it have shrunk?

3. Saturn’s average density \( \bar{\rho} \simeq 690 \text{ kg m}^{-3} \) is less than the density of water. Elementary astronomy textbooks sometimes dramatize this fact by suggesting that Saturn would float in a (very big) tub of water. Is this at all plausible? What would happen if you tried?