Solve the problems listed below, and write up your answers clearly and completely. Do not turn in rough work – instead, make a clean copy after checking your calculations. Use English sentences and phrases to explain your solution and describe key equations. Show your work!

1. Assume the Oort Cloud has a total mass $M_C \simeq 100M_E \simeq 6 \times 10^{26}$ kg and contains $N_C \simeq 10^{13}$ objects with densities of $\bar{\rho} \simeq 1000$ kg m$^{-3}$.

(a) Assuming that Oort Cloud objects are roughly spherical, what is the radius $R_o$ of an average Oort Cloud object?

(b) Suppose $N_C$ such objects are uniformly distributed through a spherical volume of radius $R_C = 10^4$ AU. What is the mean free path $\ell$ of a typical Oort Cloud object?

(c) Assume a typical Oort Cloud object has a roughly circular orbit with a semi-major axis $a = 10^4$ AU. How many orbits does it complete in $4.5 \times 10^9$ y?

(d) Using your results for parts (b) and (c) above, what is the probability that a typical Oort Cloud object has been involved in a collision since the Solar System formed?

2. Saturn’s rings have nonzero thickness because ring particle orbits are slightly inclined with respect to each other. Thus, a particle which attains a maximum height $z = h$ above the ring plane at one point on its orbit will descend to a minimum height $z = -h$ below the ring plane at another point. Saturn’s A ring has a radius of $r_A \simeq (1.30 \pm 0.07) \times 10^8$ m, and Saturn’s mass is $M_S \simeq 5.7 \times 10^{26}$ kg.

(a) Assuming for simplicity that Saturn’s gravitational field is spherical, what is period of the vertical oscillation executed by a ring particle orbiting at radius $r = r_A$?

(b) Given that Saturn’s A ring is about 30 m thick, estimate the average vertical speed of a typical ring particle.

(c) Ring particles are chunks of ice; most are a few cm in diameter. Suppose two ring particles collide at the speed you just calculated. Would you expect them to shatter, melt, stick together gravitationally, or rebound elastically?

3. A nearly-circular orbit of radius $r$ around an oblate planet of mass $M$, mean radius $R$, and second gravitational moment $J_2$ has orbital, radial, and vertical periods

$$P(r) \simeq 2\pi \sqrt{\frac{r^3}{GM}} \left(1 + \frac{3}{2} J_2 R^2 \right)^{-1/2},$$  

$$P_r(r) \simeq 2\pi \sqrt{\frac{r^3}{GM}} \left(1 - \frac{3}{2} J_2 R^2 \right)^{-1/2},$$  

$$P_z(r) \simeq 2\pi \sqrt{\frac{r^3}{GM}} \left(1 + \frac{9}{2} J_2 R^2 \right)^{-1/2},$$

respectively. (For this problem, it’s easier to think in terms of period $P$ instead of angular velocity $2\pi/P$, but equations (1) – (3) mean the same thing as the expressions for the orbital, radial, and vertical angular velocities $n(r)$, $\kappa(r)$, and $\mu(r)$ you’ve seen in class.)
(a) Jupiter has mass $M \simeq 1.90 \times 10^{27}$ kg, mean radius $R \simeq 7.0 \times 10^7$ m, and gravitational moment $J_2 \simeq 0.015$. Calculate $P$, $P_r$, and $P_z$ for Jupiter’s satellite Metis, which orbits at radius $r_{\text{met}} = 1.28 \times 10^8$ m. Verify that $P_z < P < P_r$.

(b) Metis travels exactly $360^\circ$ around Jupiter (one orbit) in time $P$. How many degrees does it travel around Jupiter in time $P_z$? The difference between this angle and $360^\circ$ tells you how much the line of nodes (intersection of orbit and equatorial planes) precesses per orbit. If the orbit is counter-clockwise, which way does the line of nodes shift?

(c) Repeat parts (a) and (b) for a hypothetical satellite orbiting at radius $r = 2r_{\text{met}}$. Measured in terms of degrees per orbit, how much faster or slower does this satellite’s orbit precess?