

Chapter 11

Shapes and Rotation of Elliptical Galaxies

An elliptical galaxy's rotation does not always correlate with its flattening. A detailed analysis based on moments of the collisionless Boltzmann equation predicts that galaxies flattened by rotation obey a simple one-to-one relationship between rotation velocity and shape. Bright E galaxies do not obey this relationship; their non-spherical shapes may instead be maintained by anisotropic random velocities. Anisotropic velocities can support triaxial equilibria, as Schwarzschild explicitly showed by numerically constructing equilibrium models of triaxial galaxies. In such galaxies the rotation axis need not be parallel to the minor axis; such kinematic misalignments are indeed observed in some ellipticals.

11.1 The Tensor Virial Theorem

The collisionless Boltzmann equation (CBE) is a partial differential equation obeyed by a function of six dimensions. We can tame this unwieldy equation by integrating along one or more dimensions; each integration literally reduces the CBE to a shadow of its former self, but for some purposes these shadows contain all the information needed to solve a particular problem.

To derive the tensor virial equation, we multiply the CBE by $v_j r_k$ and integrate over all velocities and positions (BT87, Chapter 4.3). First, we multiply by v_j and integrate over velocity:

$$\int d\mathbf{v} v_j \frac{\partial f}{\partial t} + \int d\mathbf{v} v_j v_i \frac{\partial f}{\partial r_i} - \int d\mathbf{v} v_j \frac{\partial \Phi}{\partial r_i} \frac{\partial f}{\partial v_i} = 0. \quad (11.1)$$

Using integration by parts to simplify the third term, we obtain

$$\frac{\partial}{\partial t} (\rho \overline{v_j}) + \frac{\partial}{\partial r_i} (\rho \overline{v_j v_i}) + \rho \frac{\partial \Phi}{\partial r_j} = 0. \quad (11.2)$$

where

$$\rho \equiv \int d\mathbf{v} f, \quad \rho \overline{v_i} \equiv \int d\mathbf{v} f v_i, \quad \rho \overline{v_j v_i} \equiv \int d\mathbf{v} f v_j v_i. \quad (11.3)$$

Next, multiplying each term by r_k and integrating over position yields

$$\frac{\partial}{\partial t} \int d\mathbf{r} r_k \rho \overline{v_j} = - \int d\mathbf{r} r_k \frac{\partial}{\partial r_i} (\rho \overline{v_j v_i}) - \int d\mathbf{r} r_k \rho \frac{\partial \Phi}{\partial r_j}, \quad (11.4)$$

where the time derivative has been moved outside the integral on the LHS. We will come back to this term shortly, but it is worth noting that for a system in equilibrium the integral is constant, and thus the LHS is zero. The first term on the RHS may be simplified using the divergence theorem:

$$-\int d\mathbf{r} r_k \frac{\partial}{\partial r_i} (\rho \overline{v_j v_i}) = \int d\mathbf{r} \rho \overline{v_j v_k} = 2K_{jk}. \quad (11.5)$$

Here K_{jk} is the *kinetic energy tensor*, defined as

$$K_{jk} \equiv \frac{1}{2} \int d\mathbf{r} d\mathbf{v} f(\mathbf{r}, \mathbf{v}) v_j v_k \quad (11.6)$$

Note that K_{jk} is symmetric, $K_{jk} = K_{kj}$, and that its trace is the total kinetic energy of the system.

The second term on the RHS is the *potential energy tensor*,

$$W_{jk} \equiv - \int d\mathbf{r} r_k \rho \frac{\partial \Phi}{\partial r_j}. \quad (11.7)$$

Under the assumption that the stellar mass density $\rho(\mathbf{r})$ is also the source of the gravitational field, it follows that W_{jk} is symmetric, that its trace is the potential energy U , that for a spherical system $W_{jk} = \delta_{jk} U/3$, and that for a system flattened along the z direction $W_{xx}/W_{zz} > 1$ (BT87, Chapter 2.5).

Using the symmetry of K_{jk} and W_{jk} , (11.4) becomes

$$\frac{1}{2} \frac{d}{dt} \int d\mathbf{r} \rho (r_k \overline{v_j} + r_j \overline{v_k}) = 2K_{jk} + W_{jk}. \quad (11.8)$$

The LHS, explicitly symmetrized over k and j , is one-half of the second time derivative of the *moment of inertia tensor*,

$$I_{jk} = \int d\mathbf{r} \rho r_j r_k. \quad (11.9)$$

Putting everything together finally gives the tensor virial equation,

$$\frac{1}{2} \frac{d^2}{dt^2} I_{jk} = 2K_{jk} + W_{jk}. \quad (11.10)$$

Remark: the trace of (11.10) is the more familiar scalar virial theorem; see BT87, Chapter 4.3 for a discussion.

11.2 Oblate Rotators and E Galaxies

Consider an idealized model of an oblate galaxy with density contours which are similar, concentric spheroids. The short axis of the model is aligned with the z axis of the coordinate system. From Chapter 2.5 of BT87, we know that the potential energy tensor is diagonalized in this coordinate system and that

$$W_{xx} = W_{yy}, \quad \frac{W_{xx}}{W_{zz}} = q(\varepsilon) > 1, \quad (11.11)$$

where $\varepsilon = 1 - c/a$ is the ellipticity of the model and the function $q(\varepsilon)$ may be calculated from the expressions given in Table 2.2 of BT87; a reasonably accurate approximation is

$$q(\varepsilon) \simeq 1 + \frac{\varepsilon}{2(1-\varepsilon)}. \quad (11.12)$$

The motions of stars within the model may be split into a net *streaming* motion and a *random* dispersion with respect to the streaming motion at each point. The total kinetic energy tensor is just the sum of the tensors for these separate motions. Assuming that the only streaming motion is rotation about the z axis, the associated KE tensor is

$$2K^{(s)} = \frac{1}{2}Mv_0^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (11.13)$$

where M is the total mass and v_0 is the mass-weighted rotation velocity (BT87, Chapter 4.3(b)). Allowing for the possibility that the random dispersion in the z direction is different from the dispersions in the x and y directions, the KE tensor associated with the random motion is

$$2K^{(r)} = M\sigma_0^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - \delta \end{bmatrix}, \quad (11.14)$$

where σ_0^2 is the mass-weighted random velocity in the x direction, and δ parametrizes the velocity anisotropy.

Assuming the galaxy is in equilibrium, $d^2I_{jk}/dt^2 = 0$ and the tensor virial theorem states that $2K_{jk}^{(s)} + K_{jk}^{(r)} + W_{jk} = 0$. Then

$$q(\varepsilon) = \frac{W_{xx}}{W_{zz}} = \frac{K_{xx}}{K_{zz}} = \frac{\frac{1}{2}Mv_0^2 + M\sigma_0^2}{(1 - \delta)M\sigma_0^2}, \quad (11.15)$$

which may be rearranged to give

$$\frac{v_0}{\sigma_0} = \sqrt{2(1 - \delta)q(\varepsilon) - 2} \quad (11.16)$$

This is the promised relationship between shape and kinematics. Note that this equation is *exact* for the somewhat idealized galaxy model adopted here.

The application of (11.16) is illustrated by a couple of examples:

- **Non-rotating galaxy:** if $v_0 = 0$, then the velocity anisotropy is given by

$$1 - \delta = \frac{1}{q(\varepsilon)} \simeq \frac{2 - 2\varepsilon}{2 - \varepsilon}. \quad (11.17)$$

- **Isotropic rotating galaxy:** if $\delta = 0$, then the rotation velocity is given by

$$\frac{v_0}{\sigma_0} = \sqrt{2q(\varepsilon) - 2} \simeq \sqrt{\frac{\varepsilon}{1 - \varepsilon}}. \quad (11.18)$$

The observational application of these relationships is somewhat complicated by projection effects (see BT87, Chapter 4.3(b)). However, for isotropic rotators ($\delta = 0$) projection diminishes *apparent* ellipticity and rotation velocity alike in such a way that (11.18) is still roughly correct. Thus to roughly quantify the rotational flattening of elliptical galaxies it's conventional to use the ratio

$$(v/\sigma)^* \equiv \frac{(v_{\text{los}}/\sigma_{\text{los}})}{\sqrt{2q(\varepsilon) - 2}} \simeq \frac{(v_{\text{los}}/\sigma_{\text{los}})}{\sqrt{\varepsilon/(1 - \varepsilon)}}. \quad (11.19)$$

Here v_{los} and σ_{los} are the line-of-sight rotation velocity and velocity dispersion, respectively. Note that $(v/\sigma)^* \simeq 1$ for a rotationally flattened galaxy, and < 1 for a galaxy flattened by velocity anisotropy.

The degree of rotational support depends on galaxy luminosity in a way which suggests the presence of several sub-families of elliptical galaxies (eg. BM98, Fig. 11.7). Luminous elliptical galaxies span a range of $(v/\sigma)^*$ values, but most rotate far too slowly to be flattened by rotation. Somewhat fainter ellipticals, and the bulges of disk galaxies, have $(v/\sigma)^* \simeq 1$; rotation accounts for the flattening of these systems. The faintest ellipticals, on the other hand, have low $(v/\sigma)^*$ values, indicating a significant role for velocity anisotropy.

11.3 Schwarzschild's Method

The observations of slow rotation in luminous elliptical galaxies implies that such objects are flattened by velocity anisotropy rather than rotation. Anisotropically flattened systems need not be axisymmetric; some special mechanism would be needed to make two of the three perpendicular dispersions identical. Without such a mechanism, the most plausible case is that all three axes are different; in other words, these galaxies are *triaxial*. But modeling of triaxial galaxies was difficult – it was not clear, for example, that equilibrium triaxial galaxy models could even exist!

Schwarzschild (1979) invented a powerful method for constructing both axisymmetric and triaxial models of equilibrium galaxies. The key idea is that, in an equilibrium system, each orbit generates a certain density distribution; by forming a weighted sum of the density distributions generated by all possible orbits, one should be able to recover the total density distribution of the system. Fig. 11.1 illustrates the 3-D density distributions generated by two orbits in a triaxial potential.

Schwarzschild's method has been enormously elaborated in recent years as a tool for modeling observed galaxies, but the basic procedure works as follows:

1. Specify the mass model $\rho(\mathbf{r})$ and find the corresponding potential.
2. Construct a grid of K cells in position space.
3. Chose initial conditions for a set of N orbits, and for each one,
 - (a) integrate the equations of motion for many orbital periods, and
 - (b) keep track of how much time the orbit spends in each cell, which is a measure of how much mass the orbit contributes to that cell.
4. Determine non-negative weights for each orbit such that the summed mass in each cell is equal to the mass implied by the original $\rho(\mathbf{r})$.

The resulting set of orbital weights represents the same information as does the distribution function.

Step #4 is the most subtle. Formally, let $M(c)$ be the integral of $\rho(\mathbf{r})$ over cell c , and let $P_i(c)$ be the mass contributed to cell c by orbit i . The task is then to find N non-negative quantities Q_i such that

$$M(c) = \sum_{i=1}^N Q_i P_i(c) \quad (11.20)$$

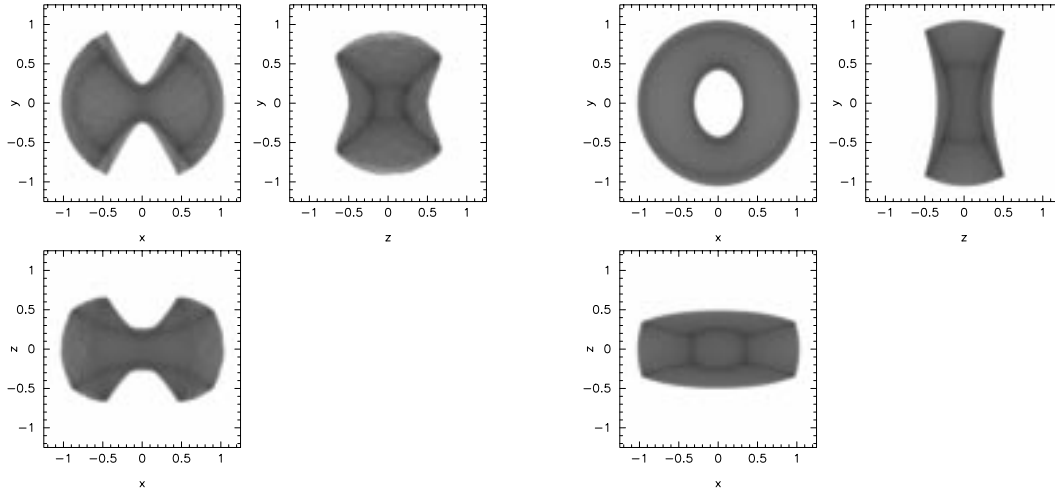


Figure 11.1: Time-averaged orbits in a triaxial logarithmic potential (6.20) with $b = 0.9$, $c = 0.8$, and $R_c = 0.2$. Left: a box orbit generated by starting at position $(x, y, z) = (1, 0, 0)$ with velocity $(v_x, v_y, v_z) = (0, 0.3, 0.4)$. Right: a minor-axis tube orbit generated by starting at position $(x, y, z) = (1, 0, 0)$ with velocity $(v_x, v_y, v_z) = (0, 0.6, 0.4)$.

simultaneously for all cells. It's generally necessary to take $N \gg K$ so as to obtain a reasonably rich set of 'basis functions'. Solutions have been found using a number of numerical techniques, including linear programming (Schwarzschild 1979), non-negative least squares (Pfenniger 1984), Lucy's method (Newton & Binney 1984), and maximum entropy (Richstone & Tremaine 1988). In general, (11.20) has *many* solutions, reflecting the fact that many *different* distribution functions are consistent with a given mass model. Some methods allow one to specify additional constraints so as to select solutions with special properties (maximum rotation, radial anisotropy, *etc.*).

By constructing an equilibrium triaxial system, Schwarzschild (1979) explicitly showed that triaxial galaxies *could* exist. He subsequently extended this result to include triaxial systems with slow figure rotation about the minor axis (Schwarzschild 1982).

11.4 Misaligned Rotators

Misaligned rotation offers compelling evidence that some elliptical galaxies are indeed triaxial. In an oblate galaxy, all orbits are minor-axis tubes; such a galaxy can only rotate about its minor axis, and a slit placed along the projected minor axis will not show any rotation.

In a triaxial galaxy, on the other hand, both minor-axis and major-axis tube orbits exist and have nonzero angular momentum. If all major-axis tubes circulate in the same direction, while equal numbers of minor-axis tubes circulate in opposite directions, the galaxy will have a net spin vector parallel to its *major* axis. Conversely, if the major-axis tubes circulate both directions, while the minor-axis tubes all circulate in same direction, the galaxy will spin parallel to its *minor* axis. Obviously, any intermediate situation between these two extremes is also possible; in such in-between cases, the spin vector lies somewhere in the plane containing the galaxy's major and minor axes.

Examples of misaligned rotation in elliptical galaxies are not hard to find. One survey of 28

galaxies found 7 exhibiting significant rotation with a slit placed along their minor axes (Wagner, Bender, & Mollenhoff 1988). Further studies show that while *most* galaxies are not strongly misaligned, a significant subset are, and a few systems exhibiting almost pure major-axis rotation (Franx, Illingworth, & de Zeeuw 1991). In many of these galaxies, the axis of rotation varies systematically with radius; for example, NGC 5128 appears to rotate about its minor axis at small radii, but the rotation axis shifts by almost 90° beyond ~ 10 kpc (Peng, Ford, & Freeman 2004).