Chapter 9

Collisionless Stellar-Dynamical Systems

A wide range of astronomical systems may be idealized as configurations of point masses interacting through gravity. But in galaxies, the effects of interactions between individual stars accumulate so gradually that they can be neglected even over timescales vastly longer than the age of the universe. This permits a simpler description, the collisionless Boltzmann equation (BT08, Ch. 4.1).

9.1 N-body Equations of Motion

Any system in which physical collisions are rare may be idealized as a collection of N *bodies*, each with position $\mathbf{r}_i$, velocity $\mathbf{v}_i$, mass m_i, and infinitesimal size. The hamiltonian for such a system is

$$H(\{\mathbf{r}_i\},\{\mathbf{v}_i\}) = \sum_i \frac{1}{2} m_i |\mathbf{v}_i|^2 - \sum_i \sum_{j<i} \frac{G m_i m_j}{|\mathbf{r}_j - \mathbf{r}_i|}, \tag{9.1}$$

where H depends on all body positions and velocities, the first sum runs over all N bodies, the second runs over all *pairs* of bodies, and G is the gravitational constant. Then the equations of motion are

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \qquad \frac{d\mathbf{v}_i}{dt} = \sum_{j \neq i} \frac{G m_j (\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3}, \tag{9.2}$$

where the sum runs over all bodies except body i.

N-body systems obey several basic conservation laws. In BT08 (Appendix D.2) these laws are derived by manipulating (9.2). However, they may also be recognized directly from the form of the hamiltonian; Noether's theorem states that each *symmetry* of H gives rise to a conservation law. A symmetry is a transformation which leaves the physical system unchanged. For example, translation in time, $t \rightarrow t + \Delta t$, is a symmetry of (9.1) because H is not an explicit function of time; consequently the total system energy $E = T + U = H$ is conserved. Likewise, symmetry with respect to translation in space, $\mathbf{r} \rightarrow \mathbf{r} + \Delta\mathbf{r}$, implies conservation of total linear momentum, and symmetry with respect to rotation gives rise to conservation of total angular momentum.

9.2 Virial Parameters

Another general result shown by manipulating (9.2) is the *scalar virial theorem* (BT08, Chapter 7.2.1), which states that for a system in equilibrium,

$$2\langle T\rangle + \langle U\rangle = 0\,, \tag{9.3}$$

where T and U are the total kinetic and potential energy, respectively, and the angle-brackets indicate time-averages. Since $E = T + U$, the time-averaged kinetic and potential energies are related to the conserved total energy by

$$\langle T\rangle = -E\,, \qquad \langle U\rangle = 2E\,. \tag{9.4}$$

The total mass M and total energy E of an N-body system thus define characteristic velocity and length scales

$$V_\mathrm{V}^2 = 2\frac{\langle T\rangle}{M} = 2\frac{|E|}{M}\,, \qquad R_\mathrm{V} = -G\frac{M^2}{\langle U\rangle} = G\frac{M^2}{2|E|}\,. \tag{9.5}$$

These are sometimes known as the *virial* velocity and radius, respectively.

The quantity $t_\mathrm{c} = R_\mathrm{V}/V_\mathrm{V}$ is an estimate of the time a typical body takes to cross the system. This timescale may be expressed in several different ways; for example, in terms of the total mass M and energy E, it is

$$t_\mathrm{c} = G\sqrt{M^5/8|E|^3}\,. \tag{9.6}$$

Note that M and E are conserved, so t_c is a constant even for systems which are far from dynamical equilibrium. In such cases t_c approximates the time-scale over which the system evolves toward equilibrium.

Another expression for t_c follows from the substitution $V_\mathrm{V}^2 = GM/R_\mathrm{V}$ valid for systems near equilibrium:

$$t_\mathrm{c} = (GM/R_\mathrm{V}^3)^{-1/2}\,. \tag{9.7}$$

Here the quantity M/R_V^3, which has units of density, appears. In systems with galaxy-like density profiles, the virial radius is approximately proportional to the half-mass radius: $R_\mathrm{h} \simeq 0.4R_\mathrm{V}$. Using this relationship, it follows that

$$t_\mathrm{c} \simeq 1.36(G\rho_\mathrm{h})^{-1/2}\,, \tag{9.8}$$

where ρ_h is the mean density within R_h. Since the crossing time is just supposed to indicate a typical time-scale for orbital motion, it is OK to drop the numerical constant, and define

$$t_\mathrm{c} \equiv (G\rho_\mathrm{h})^{-1/2}\,. \tag{9.9}$$

9.3 Relaxation Time

Consider an encounter with impact parameter b and velocity v between two stars of mass m, shown in Fig. 9.1. Using the impulse approximation, the transverse velocity acquired is

$$\delta v_\mathrm{t} = \frac{2Gm}{bv}\,. \tag{9.10}$$

The impulse approximation is well-justified in systems with large N because large-angle deflections are very rare; the impact parameter leading to a large deflection is

$$b_\mathrm{min} = Gm/v^2 \simeq R_\mathrm{V}/N\,, \tag{9.11}$$

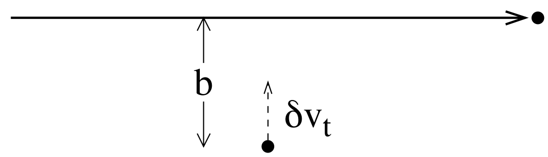

Figure 9.1: An impulsive passage; as a result of the gravitational pull of the passing star at top, the test stars acquires velocity δv_{t}.

where the second equality follows by using the virial theorem and assuming that $v \simeq V_{\mathrm{V}}$ and all stars have mass m. This is much smaller than the typical distance between particles, which is of order $R_{\mathrm{V}}/N^{1/3}$. In a typical stellar-dynamical system there is roughly one close encounter per crossing time, regardless of N.

A key assumption made in considering the effects of interactions between individual stars is that *encounters are not correlated* with one another; thus collective effects are neglected. This assumption works well in many cases, though examples of collective relaxation will come up later in this course. If each encounter is uncorrelated with the last, the cumulative effect of many encounters is a *random walk* in velocity; perturbations add in quadrature. During a single passage through the system, a typical star has roughly

$$dn = \frac{N}{\pi R_{\mathrm{V}}^2} 2\pi b\,db \tag{9.12}$$

encounters with impact parameters between b and $b + db$. Here the first factor is just the surface density of stars, and the second factor is the area of an annulus with radius b and width db. Adding velocity perturbations in quadrature, the deflection due to these dn encounters is

$$dv^2 = \delta v_{\mathrm{t}}^2 dn = 8N \left(\frac{Gm}{R_{\mathrm{V}} v}\right)^2 \frac{db}{b}, \tag{9.13}$$

and the total velocity perturbation acquired in one crossing time is

$$\Delta v^2 = \int_{b_{\min}}^{R_{\mathrm{V}}} dv^2 = 8N \left(\frac{Gm}{R_{\mathrm{V}} v}\right)^2 \ln\left(\frac{R_{\mathrm{V}}}{b_{\min}}\right). \tag{9.14}$$

Here the logarithmic factor arises from the integration over impact parameter from $b_{\min}$ to R_{V}; each decade between $b_{\min}$ and R_{V} contributes equally to the total deflection. Thus, even though a single wide encounter, with $b \gg b_{\min}$, scarcely perturbs the star, the *cumulative* effect of such encounters typically dominates the evolution of the system!

Finally, estimating the encounter velocity v from the virial velocity $V_{\mathrm{V}} \simeq \sqrt{GNm/R_{\mathrm{V}}}$ gives

$$\Delta v^2 = \frac{8 \ln N}{N} V_{\mathrm{V}}^2 \tag{9.15}$$

for the total change in a typical star's velocity per crossing time t_{c}.

The **relaxation time** is the time over which the cumulative effect of stellar encounters becomes comparable to a star's initial velocity. From (9.15) this is

$$t_{\mathrm{r}} = \frac{V_{\mathrm{V}}^2}{\Delta v^2} t_{\mathrm{c}} \simeq \frac{N}{8 \ln N} t_{\mathrm{c}}. \tag{9.16}$$

In stellar systems with large N this time is much longer than the crossing time; the evolution of such systems proceeds on two widely-separated timescales. Relaxation due to stellar encounters plays an important role in the evolution of star clusters. But a typical galaxy has 10^{11} stars but is less than 100 crossing times old, so the cumulative effects of encounters between stars are pretty insignificant. This justifies the next step, which is to idealize a galaxy as a continuous mass distribution, effectively taking the limit $t_{\mathrm{r}} \to \infty$.

9.4 Collisionless Dynamics

In the continuum limit, each star moves in the smooth gravitational field $\Phi(\mathbf{r},t)$ of the galaxy. Thus instead of thinking about motion in a phase space of $6N$ dimensions, we can think about motion in a phase space of just 6 dimensions. This is a vast simplification!

9.4.1 Distribution function

Rather than keeping track of individual stars, a galaxy may be described by the one-body distribution function; let

$$f(\mathbf{r},\mathbf{v},t)d^3\mathbf{r}d^3\mathbf{v} \tag{9.17}$$

be the mass of stars in the phase-space volume $d^3\mathbf{r}d^3\mathbf{v}$ at $(\mathbf{r},\mathbf{v})$ and time t. This provides a statistically complete description if stars are uncorrelated, as assumed above.

9.4.2 Collisionless Boltzmann equation

The motion of matter in phase space is governed by the phase-flow,

$$(\dot{\mathbf{r}},\dot{\mathbf{v}}) = (\mathbf{v},-\nabla\Phi)\,. \tag{9.18}$$

How does this affect the total amount of mass in the phase space volume $d^3\mathbf{r}d^3\mathbf{v}$? Consider the 2-D example shown in Fig. 9.2, where a cell of volume $2\delta r \times 2\delta v$ is located at (r_0, v_0). The mass within the cell is

$$4\delta r\delta v f(r_0,v_0,t)\,. \tag{9.19}$$

Matter flows in to the cell through the left and top sides, and out through the right and bottom sides; the rate of change of the mass within the cell is

$$\begin{aligned} 2\delta v &\left[f(r_0-\delta r,v_0)\dot{r}|_{r_0-\delta r,v_0} - f(r_0+\delta r,v_0)\dot{r}|_{r_0+\delta r,v_0}\right] + \\ 2\delta r &\left[f(r_0,v_0-\delta v)\dot{v}|_{r_0,v_0-\delta v} - f(r_0,v_0+\delta v)\dot{v}|_{r_0,v_0+\delta v}\right] \\ &\simeq -4\delta r\delta v\left[\frac{\partial}{\partial r}(f\dot{r}) + \frac{\partial}{\partial v}(f\dot{v})\right]_{r_0,v_0}\,. \end{aligned} \tag{9.20}$$

Here the first line is the difference between the mass flowing in at the left and out at the right, the second line is the difference between the mass flowing in at the top and out at the bottom, and the third line follows from recognizing these differences as derivatives. Equating the rate of change to the time derivative of the mass within the cell yields

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial r}(f\dot{r}) + \frac{\partial}{\partial v}(f\dot{v}) = 0\,, \tag{9.21}$$

which is the equation of continuity.

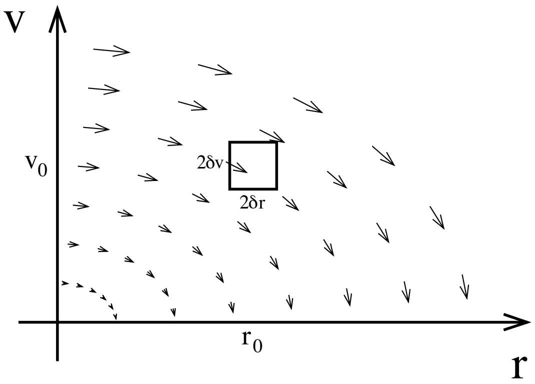

Figure 9.2: Phase space cell of volume $4\delta r\delta v$ located at $(\mathbf{r}_0, \mathbf{v}_0)$. The arrows represent the phase-space flow field $(\dot{\mathbf{r}}, \dot{\mathbf{v}}) = (\mathbf{v}, -\nabla\Phi)$, which transports matter in to and out of the cell.

The analogous continuity equation for a 6-D phase-space is

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{r}}(f\dot{\mathbf{r}}) + \frac{\partial}{\partial \mathbf{v}}(f\dot{\mathbf{v}}) = 0\,. \tag{9.22}$$

Using (9.18) for the phase-flow yields the collisionless Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} = 0\,. \tag{9.23}$$

The collisionless Boltzmann equation or CBE describes the evolution of the distribution function $f(\mathbf{r}, \mathbf{v}, t)$. It contains Newton's $\mathbf{F} = m\mathbf{a}$ [via (9.18)] as well as conservation of matter, and therefore serves as the fundamental equation of galactic dynamics.

In a galaxy we often deal with several distinct kinds of collisionless matter; for example, stars and dark matter[1]. We can define separate distribution functions f_{stars} and f_{dark} to describe these components; each of these obeys (9.23), and their sum does so as well. We can also, for example, define separate distribution functions for the disk and bulge of a spiral galaxy, or even a distribution function for a particular class of stars – although in that case the right-hand side of (9.23) may be nonzero due to processes of stellar evolution! On the other hand, interstellar matter is *not* collisionless and its dynamical evolution does not obey (9.23).

9.4.3 Gravity

The gravitational field $\Phi(\mathbf{r}, t)$ is given by Poisson's equation,

$$\nabla^2\Phi = 4\pi G \int d^3\mathbf{v}\, f(\mathbf{r}, \mathbf{v}, t)\,, \tag{9.24}$$

where the integral is taken over all velocities. Together, (9.23) and (9.24) may be viewed as a pair of coupled PDEs which describe the dynamical evolution of a galaxy.

[1] Some theorists have suggested that dark matter might interact collisionally *with itself*, but there's no compelling evidence for this hypothesis.

9.4.4 Conservation of phase space density

Let $(\mathbf{r},\mathbf{v}) = (\mathbf{r}(t),\mathbf{v}(t))$ be the orbit of a star. What is the rate of change of $f(\mathbf{r},\mathbf{v},t)$ along the star's orbit? The answer is

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \dot{\mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{r}} + \dot{\mathbf{v}} \cdot \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0\,, \tag{9.25}$$

where the first equality is just the definition of the convective derivative in phase-space, the second equality follows on substituting the phase-flow (9.18), and the last equality follows from the CBE (9.23). Thus, *phase-space density is conserved along every orbit*.

This fundamental and completely general result shows that the CBE has a much greater level of symmetry than the N-body equations of motion; whereas the latter conserves a fairly small set of parameters, the CBE conserves $f(\mathbf{r},\mathbf{v},t)$ along an *infinite* number of stellar orbits. We can take advantage of this infinite array of conservation laws to obtain some important results even when we can't explicitly solve the CBE.

Problems

7.1. Using the definition of the large-angle impact parameter $b_{\rm min}$, show that to order of magnitude a typical star has a $1/N$ chance of a large-angle deflection per crossing time.

7.2. Assuming all these systems are near equilibrium, estimate the relaxation time $t_{\rm r}$ for

1. an open star cluster with $N = 10^3$ and $R_{\rm h} = 2\,{\rm pc}$;
2. a globular star cluster with $N = 10^6$ and $R_{\rm h} = 10\,{\rm pc}$;
3. an elliptical galaxy with $N = 10^{11}$ and $R_{\rm h} = 3\,{\rm kpc}$;
4. a galaxy cluster with $N = 10^3$ and $R_{\rm h} = 0.5\,{\rm Mpc}$;