

## Chapter 7

# Gravitational Fields

Analytic calculation of gravitational fields is easy for spherical systems, but few *general* results are available for non-spherical mass distributions. Mass models with tractable potentials may however be combined to approximate the potentials of real galaxies.

### 7.1 Conservative Force Fields

In a one-dimensional system it is always possible to define a potential energy corresponding to any given  $f(x)$ ; let

$$U(x) = - \int_{x_0}^x dx' f(x'), \quad (7.1)$$

where  $x_0$  is an arbitrary position at which  $U = 0$ . Different choices of  $x_0$  produce potential energies differing by an additive constant; this constant has no influence on the dynamics of the system.

In a space of  $n > 1$  dimensions the analogous path integral,

$$U(\mathbf{x}) = - \int_{\mathbf{x}_0}^{\mathbf{x}} d^n \mathbf{x}' \cdot \mathbf{f}(\mathbf{x}'), \quad (7.2)$$

may depend on the exact route taken from points  $\mathbf{x}_0$  to  $\mathbf{x}$ ; if it does, a unique potential energy cannot be defined. One condition for this integral to be path-independent is that the integral of the force  $\mathbf{f}(\mathbf{x})$  around all closed paths vanishes. An equivalent condition is that there is some function  $U(\mathbf{x})$  such that

$$\mathbf{f}(\mathbf{x}) = -\nabla U. \quad (7.3)$$

Force fields obeying these conditions are *conservative*. The gravitational field of a stationary point mass is an example of a conservative field; the energy released in moving from radius  $r_1$  to radius  $r_2 < r_1$  is *exactly* equal to that consumed in moving back from  $r_2$  to  $r_1$ .

### 7.2 General Results

In astrophysical applications it's natural to work with the path integral of the *acceleration* rather than the force; this integral is the potential energy per unit mass or **gravitational potential**,  $\Phi(\mathbf{x})$ ,

and the potential energy of a test mass  $m$  is just  $U = m\Phi(\mathbf{x})$ . For an arbitrary mass density  $\rho(\mathbf{x})$ , the potential is

$$\Phi(\mathbf{x}) = -G \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \quad (7.4)$$

where  $G$  is the gravitational constant and the integral is taken over all space.

**Poisson's equation** provides another way to express the relationship between density and potential:

$$\nabla^2\Phi = 4\pi G\rho. \quad (7.5)$$

Note that these relationships are *linear*; if  $\rho_1$  generates  $\Phi_1$  and  $\rho_2$  generates  $\Phi_2$  then  $\rho_1 + \rho_2$  generates  $\Phi_1 + \Phi_2$ .

**Gauss's theorem** relates the mass within some volume  $V$  to the gradient of the field on its surface  $dV$ :

$$4\pi G \int_V d^3\mathbf{x} \rho(\mathbf{x}) = \int_{dV} d^2\mathbf{S} \cdot \nabla\Phi, \quad (7.6)$$

where  $d^2\mathbf{S}$  is an element of surface area with an outward-pointing normal vector.

## 7.3 Spherical Potentials

Consider a spherical shell of mass  $m$ ; Newton's first and second theorems (BT87, Ch. 2.1) imply

- the acceleration inside the shell vanishes, and
- the acceleration outside the shell is  $-Gm/r^2$ .

From this, it follows that the potential of an *arbitrary* spherical mass distribution is

$$\Phi(r) = - \int_{r_0}^r dx a(x) = G \int_{r_0}^r dx \frac{M(x)}{x^2}, \quad (7.7)$$

where the enclosed mass is

$$M(r) = 4\pi \int_0^r dx x^2 \rho(x). \quad (7.8)$$

If  $M$  converges for  $r \rightarrow \infty$ , it's convenient to set  $r_0 = \infty$ ; then  $\Phi(r) < 0$  for all finite  $r$ .

### 7.3.1 Elementary Examples

A **point** of mass  $M$  has the potential

$$\Phi(r) = -G \frac{M}{r}. \quad (7.9)$$

This is known as a *Keplerian* potential since orbits in this potential obey Kepler's three laws. A circular orbit at radius  $r$  has velocity  $v_c(r) = \sqrt{GM/r}$ .

A **uniform sphere** of mass  $M$  and radius  $a$  has the potential

$$\Phi(r) = \begin{cases} -2\pi G\rho(a^2 - r^2/3), & r < a \\ -GM/r, & r > a \end{cases} \quad (7.10)$$

where  $\rho = M/(4\pi a^3/3)$  is the mass density. Outside the sphere the potential is Keplerian, while inside it has the form of a parabola; both the potential and its derivative are continuous at the surface of the sphere.

A **singular isothermal sphere** with density profile  $\rho(r) = \rho_0(r/r_0)^{-2}$  has the potential

$$\Phi(r) = 4\pi G \rho_0 r_0^2 \ln(r/r_0). \quad (7.11)$$

The circular velocity  $v_c = \sqrt{4\pi G \rho_0 r_0^2}$  is constant with radius. This potential is often used to approximate the potentials of galaxies with flat rotation curves, but some outer cut-off must be imposed to obtain a finite total mass.

### 7.3.2 Potential-Density Pairs

Pairs of functions related by Poisson's equation provide convenient building-blocks for galaxy models. Three such functions often used in the literature are listed here; all describe models characterized by a total mass  $M$  and a length scale  $a$ :

| Name      | $\rho(r)$   | $\Phi(r)$  |
|-----------|---|--|
| Plummer   | $\frac{3M}{4\pi a^3} \left(1 + \frac{r^2}{a^2}\right)^{-5/2}$       | $\frac{-GM}{\sqrt{r^2 + a^2}}$   |
| Hernquist | $\frac{M}{2\pi} \frac{a}{r(r+a)^3}$                                 | $\frac{-GM}{r+a}$  |
| Jaffe     | $\frac{M}{4\pi} \frac{a}{r^2(r+a)^2}$                               | $\frac{GM}{a} \ln\left(\frac{a}{r+a}\right)$   |
| Gamma     | $\frac{(3-\gamma)M}{4\pi a^3} \frac{a^4}{r^\gamma(r+a)^{4-\gamma}}$ | $\frac{GM}{a} \begin{cases} \frac{1}{\gamma-2} \left[1 - \left(\frac{r}{r+a}\right)^{2-\gamma}\right], & \gamma \neq 2 \\ \ln\left(\frac{r}{r+a}\right), & \gamma = 2 \end{cases}$ |

The Plummer (1911) density profile has a finite-density core and falls off as  $r^{-5}$  at large radii; this is a steeper fall-off than is generally seen in galaxies. Hernquist (1990) and Jaffe (1983) models, on the other hand, both decline like  $r^{-4}$  at large radii; this power law has a sound theoretical basis in the mechanics of violent relaxation. The Hernquist model has a gentle power-law cusp at small radii, while the Jaffe model has a steeper cusp. Gamma models (Dehnen 1993; Tremaine et al. 1994) include both Hernquist ( $\gamma = 1$ ) and Jaffe ( $\gamma = 2$ ) models as special cases; the best approximation to the de Vaucouleurs profile has  $\gamma = 3/2$ .

## 7.4 Axisymmetric Potentials

If the mass distribution is a function of two variables, cylindrical radius  $R$  and height  $z$ , the problem of calculating the potential becomes a good deal harder. A general expression exists for infinitely thin disks, but only special cases are known for systems with finite thickness.

### 7.4.1 Thin disks

An axisymmetric disk is described by a surface mass density  $\Sigma(R)$ . Potential-surface density expressions for several important cases are collected here:

| Name   | $\Sigma(R)$  | $\Phi(R, z)$   |
|--------|--|--|
| Kuzmin | $\frac{aM}{2\pi(R^2 + a^2)^{3/2}}$                         | $\frac{-GM}{\sqrt{R^2 + (a +  z )^2}}$                   |
| Toomre | $\left(\frac{d}{da^2}\right)^{n-1} \Sigma_{\text{Kuzmin}}$ | $\left(\frac{d}{da^2}\right)^{n-1} \Phi_{\text{Kuzmin}}$ |
| Bessel | $\frac{k}{2\pi G} J_0(kR)$                                 | $\exp(-k z ) J_0(kR)$                                    |

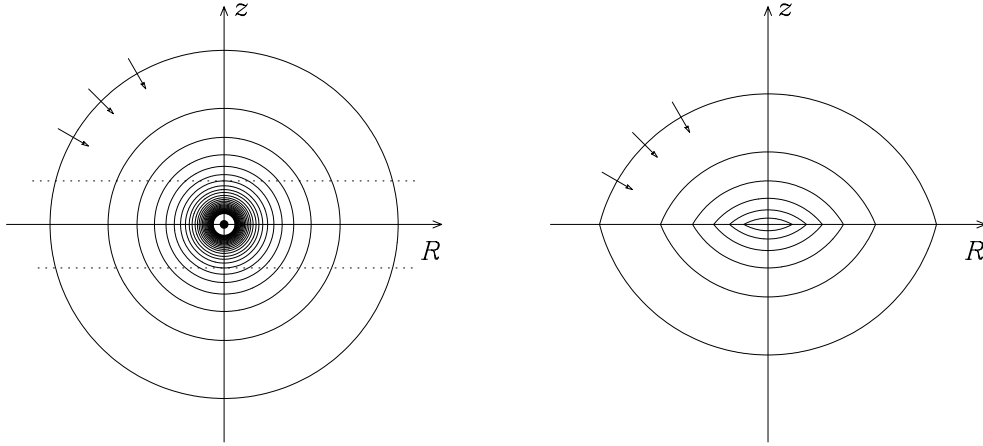


Figure 7.1: Surgery on the potential of a point mass produces the potential of a Kuzmin disk. Left: potential of a point mass. Contours show equal steps of  $\Phi \propto 1/r$ , while the arrows in the upper left quadrant show the radial force field. Dotted lines show  $z = \pm a$ . Right: potential of a Kuzmin disk, produced by excising the region  $|z| \leq a$  from the field shown on the left. Arrows again indicate the force field; note that these no longer converge on the origin.

The potential of a Kuzmin (1956) disk may be constructed by performing surgery on the field of a point of mass  $M$  (7.9), as shown in Fig. 7.1. Let the  $z$  coordinate be the axis of the disk, and join the field for  $z > a > 0$  directly to the field for  $z < -a$ . The result satisfies Laplace's equation,  $\nabla^2\Phi = 0$ , everywhere  $z \neq 0$ ; thus the matter density which generates this potential is confined to the disk plane  $z = 0$ . Gauss's theorem (7.6) can be used to find the required surface density. Consider an infinitesimal box which straddles the disk plane at radius  $R$ , and let the top and bottom surfaces of this box have area  $\delta A$ . The integral of  $\nabla\Phi$  over the surface of the box is  $2GMa\delta A/(R^2 + a^2)^{3/2}$ , and mass within the box is  $\delta A\Sigma(R)$ . Setting these two integrals equal yields the surface density  $\Sigma_K$  of Kuzmin's disk.

Toomre (1962) devised an infinite sequence of models, parameterized by the integer  $n$ . The  $n = 1$  Toomre model is identical to Kuzmin's disk, while the  $n = 2$  model is found by differentiating the potential and surface density of Kuzmin's disk by the parameter  $a^2$ ; the linearity of Poisson's equation (7.5) guarantees that the result is a valid potential-surface density pair. In the limit  $n \rightarrow \infty$  the surface density is a gaussian in  $R$ .

The 'Bessel' potential-density pair, also due to Toomre (1962), describes another field which obeys Laplace's equation everywhere except on the disk plane. Unlike other cases, the surface density – once again obtained using Gauss's theorem – is not positive-definite. What good is it? A disk with arbitrary surface density  $\Sigma(R)$  may be expanded as

$$\Sigma(R) = \int dk S(k) k J_0(kR) \quad (7.12)$$

where  $S(k)$  is the Hankel transform of  $\Sigma(R)$ ,

$$S(k) = \int dR \Sigma(R) R J_0(kR). \quad (7.13)$$

The potential generated by this disk is then

$$\Phi(R, z) = -2\pi G \int dk \exp(-k|z|) J_0(kR) S(k). \quad (7.14)$$

This method may be used to evaluate the potential of an exponential disk. Consider a disk with surface density

$$\Sigma(R) = \frac{\alpha^2 M}{2\pi} e^{-\alpha R} = \Sigma_0 e^{-\alpha R}, \quad (7.15)$$

where  $M$  is the total disk mass and  $\alpha$  is the inverse of the disk scale length. The resulting potential is

$$\Phi(R, z) = -\frac{2\pi G \Sigma_0}{\alpha^2} \int_0^\infty dk \frac{J_0(kR) e^{-k|z|}}{[1 + k^2 \alpha^{-2}]}. \quad (7.16)$$

In the plane of the disk,  $z = 0$  and this integral becomes

$$\Phi(R, 0) = -\pi G \Sigma_0 R [I_0(y) K_1(y) - I_1(y) K_0(y)], \quad (7.17)$$

where  $y = \alpha R/2$ , and  $I_n$  and  $K_n$  are modified Bessel functions of the first and second kind.

### 7.4.2 Flattened Systems

Potential-density pairs for a few flattened systems are listed here:

| Name           | $\rho(R, z)$  | $\Phi(R, z)$                                    |
|----------------|---|---|
| Miyamoto-Nagai | $\left(\frac{b^2 M}{4\pi}\right) \frac{aR^2 + (a+3\sqrt{z^2+b^2})(a+\sqrt{z^2+b^2})}{[R^2 + (a+\sqrt{z^2+b^2})^2]^{5/2} (z^2+b^2)^{3/2}}$ | $\frac{-GM}{\sqrt{R^2 + (a+\sqrt{b^2+z^2})^2}}$ |
| Satoh          | $\left(\frac{d}{db^2}\right)^n \Phi_{MN}$   | $\left(\frac{d}{db^2}\right)^n \Phi_{MN}$       |
| Logarithmic    | $\left(\frac{v_0^2}{4\pi G q^2}\right) \frac{(2q^2+1)R_c^2 + R^2 + (2-q^{-2})z^2}{(R_c^2 + R^2 + z^2 q^{-2})^2}$                          | $\frac{1}{2} v_0^2 \ln(R_c^2 + R^2 + z^2/q^2)$  |

The Miyamoto & Nagai (1975) potential includes the Plummer model (if  $a = 0$ ) and the Kuzmin disk (if  $b = 0$ ); thus it can describe a wide range of shapes, from a spherical system to an infinitely thin disk. Satoh's (1980) models are derived from the Miyamoto-Nagai model by differentiating with respect to the parameter  $b^2$ , much as Toomre's models are derived from Kuzmin's.

The logarithmic potential is often used to describe galaxies with approximately flat rotation curves; in the  $z = 0$  plane, the circular velocity  $v_c = v_0 R / \sqrt{R_c^2 + R^2}$  rises linearly for  $R \ll R_c$  and levels off for large  $R$ . The corresponding density can be found by inserting the potential in Poisson's equation. As Fig. 2.8 of BT87 shows, this density is 'dimpled' at the poles; if  $q^2 < 1/2$  the density must actually be *negative* along the  $z$  axis, which is unphysical. This dimpling is demanded by the assumption that  $\Phi(R, z)$  is stratified on surfaces of constant ellipticity as  $R \rightarrow \infty$ . In models where the potential becomes more spherical at larger radii this dimpling isn't necessary.

A great deal of analytic machinery exists to calculate potentials for systems with densities stratified on ellipsoidal surfaces (BT87, Ch. 2.3). Much of this machinery generalizes to triaxial systems; a few key results are given below.

## 7.5 Triaxial Potentials

### 7.5.1 Ellipsoidal systems

The triaxial generalization of an infinitely thin spherical shell is an infinitely thin *homoeoid*, which has constant density between surfaces  $m^2$  and  $m^2 + dm^2$ , where

$$m^2 = x^2 + \frac{y^2}{b^2} + \frac{z^2}{c^2}. \quad (7.18)$$

Just as in the spherical case, the acceleration inside the shell vanishes; thus  $\Phi = \text{const.}$  inside the shell and on its surface. Outside the shell, the potential is stratified on ellipsoidal surfaces defined by

$$m^2 = \frac{x^2}{1+\tau} + \frac{y^2}{b^2+\tau} + \frac{z^2}{c^2+\tau} \quad (7.19)$$

where the parameter  $\tau > 0$  labels the surface. Notice that for  $\tau = 0$  the isopotential surface coincides with the surface of the homoeoid, while in the limit  $\tau \rightarrow \infty$  the isopotential surfaces become spherical.

A triaxial mass distribution with  $\rho = \rho(m^2)$  is equivalent to the superposition of a series of thin homoeoids. The acceleration at a given point  $(x_0, y_0, z_0)$  is generated entirely by the mass within  $m^2 < m_0^2 = x_0^2 + y_0^2/b^2 + z_0^2/c^2$ , just as in a spherical system the acceleration at  $r$  is due only to the mass within  $r$ . Since the isopotential surfaces of a homoeoid become spherical at large  $r$ , mass lying within  $m^2 \ll m_0$  generates a nearly radial force field, while mass lying only slightly within  $m_0^2$  generates a more non-radial force field. Thus the potential of a centrally concentrated body is always more spherical than the potential of uniform body with the same axial ratios.

### 7.5.2 Logarithmic potential

The triaxial version of the logarithmic potential is

$$\Phi(x, y, z) = \frac{1}{2} v_0^2 \ln \left( R_c^2 + x^2 + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right). \quad (7.20)$$

As in the 2-D case above, the corresponding density is not guaranteed to be positive if the potential is too strongly flattened. Nonetheless, the logarithmic potential is a useful approximation when studying orbits in triaxial galaxies.

## Problems

**7.1.** Evil space aliens compress the Earth to an infinitely thin disk of uniform surface density  $\Sigma_{\oplus}$  without changing its radius. What is the gravitational acceleration experienced by a survivor standing upright at the center of this disk?

**7.2.** Figure 2-17 of BT87 compares circular-speed curves  $v_c(r)$  for three mass distributions: “an exponential disk (full curve); a point with the same total mass (dotted curve); the spherical body for which  $M(r)$  is [identical to  $M(r)$  for the exponential disk] (dashed curve)” (see BT87, p. 78). In qualitative terms, first explain why the dashed curve lies *below* the dotted curve at all radii. Then, explain why the full curve lies *above* the dotted curve at *some* radii.