Chapter 14

Local Stability of Disk Galaxies

The existence of an equilibrium solution to the CBE does not assure its stability. Real stellar systems are subject to perturbations, and if these grow they may completely transform the initial equilibrium. Thus it is critical to check that our galaxy models are actually stable.

14.1 Physics of the Jeans Instability

The Jeans (1929) instability is probably the most basic instability in gravitating systems. It was first derived to discuss a long-abandoned model for the formation of the Solar System, but it is now recognized as of fundamental importance for cosmology.

To understand the Jeans instability, consider a nearly-uniform distribution of stars containing a *slightly* overdense spherical region with radius L and density ρ. *The overdense region will collapse if the random velocities of stars are not large enough to carry them out of the region before the collapse can occur* (e.g. Jeans 1929, Toomre 1964).

The collapse timescale can be estimated by considering a star at rest on the edge of the sphere. The gravitational acceleration of this star is just GM/L^2, where the mass of the sphere is

$$M = \frac{4\pi}{3} L^3 \rho \,. \tag{14.1}$$

Then the time for this star to reach the center is just half the Keplerian period for an orbit with semimajor axis $L/2$ about a point-mass M, or

$$t_{\rm coll} = \frac{1}{2} 2\pi \frac{(L/2)^{3/2}}{\sqrt{GM}} = \sqrt{\frac{3\pi}{32 G\rho}} \,. \tag{14.2}$$

The timescale for stars to escape the overdense region is of order L divided by the *r.m.s.* stellar velocity, or

$$t_{\rm esc} = \frac{L}{(\overline{v^2})^{1/2}} \,. \tag{14.3}$$

Notice that $t_{\rm coll}$ is independent of L, while $t_{\rm esc}$ increases linearly with L. Thus small regions have $t_{\rm esc} < t_{\rm coll}$ and are stable, while large regions have $t_{\rm esc} > t_{\rm coll}$ and are unstable. The critical radius where collapse is just possible can be estimated by setting $t_{\rm esc} = t_{\rm coll}$; the result is the *Jeans length*,

$$L_{\rm J} = \sqrt{\frac{3\pi}{32} \frac{\overline{v^2}}{G\rho}} \,. \tag{14.4}$$

Only overdense regions with sizes $L > L_J$ are subject to the Jeans instability; smaller regions are obliterated by random stellar motions before they have a chance to collapse.

14.2 Physics of Disk Instabilities

To form a physical understanding of disk instabilities, we can first consider the Jeans instability in a non-rotating disk, and then consider separately the effects of rotation (Toomre 1964, GKvdK89, Ch. 9.5).

14.2.1 Nonrotating disks

A 2-D version of the Jeans instability serves as a model for local gravitational instability in a stellar disk. Let Σ be the surface density of the disk, and suppose that there is a region, of radius L and mass

$$M = \pi L^2 \Sigma, \tag{14.5}$$

which is slightly overdense. Approximating the collapse time by the same Keplerian formula as above, we have

$$t_{\rm coll} = \sqrt{\frac{\pi}{8} \frac{L}{G\Sigma}}, \tag{14.6}$$

while the escape time is again given by (14.3). Notice that $t_{\rm coll}$ is now proportional to $\sqrt{L}$; because this is still less than the linear proportionality of $t_{\rm esc}$, the reasoning used in the 3-D case applies here too. Thus by setting $t_{\rm esc} = t_{\rm coll}$ we obtain the Jeans length in 2-D

$$L_{\rm J} = \frac{\pi}{8} \frac{\overline{v^2}}{G\Sigma}. \tag{14.7}$$

As before, only overdense regions with $L > L_J$ can collapse before they are erased by random motions of stars.

14.2.2 Rotating disks

In a differentially rotating disk the local angular velocity is Oort's constant B. A circular region collapsing from radius L to radius L_1 will conserve its angular momentum, so its angular velocity is

$$\Omega_1 = B \frac{L^2}{L_1^2}. \tag{14.8}$$

If we analyze the motion of the region in a frame of reference rotating with angular velocity Ω_1 we must include an outward-directed pseudo-acceleration (centrifugal force), which at the edge of the disk is

$$a_{\rm r} = L_1 \Omega_1^2 = B^2 \frac{L^4}{L_1^3}. \tag{14.9}$$

There is also an inward acceleration due to gravity:

$$a_{\rm g} = -\frac{GM}{L_1^2}, \tag{14.10}$$

where once again a point-mass approximation has been used. Now the key idea is that *collapse will not occur if a_{r} initially increases faster than a_{g}* (Toomre 1964). This establishes a *maximum* radius for collapse, because rotation is more important on larger scales. Setting

$$\left.\frac{da_{\mathrm{r}}}{dL_1}\right|_{L_1=L} = -\left.\frac{da_{\mathrm{g}}}{dL_1}\right|_{L_1=L} \tag{14.11}$$

and solving for L yields the maximum unstable radius

$$L_{\mathrm{rot}} = \frac{2\pi}{3}\frac{G\Sigma}{B^2}\,. \tag{14.12}$$

Rotation prevents collapse on scales $L > L_{\mathrm{rot}}$.

14.2.3 Putting it all together

Combining the above results, we conclude that only regions with radii satisfying $L_{\mathrm{J}} < L < L_{\mathrm{rot}}$ can collapse; smaller scales are stabilized by random motion, while larger scales are stabilized by rotation. Thus *a disk is locally stable if* $L_{\mathrm{J}} > L_{\mathrm{rot}}$ (Toomre 1964); imposing this inequality and solving for the *r.m.s.* stellar velocity yields

$$(\overline{v^2})^{1/2} > \frac{4}{\sqrt{3}}\frac{G\Sigma}{B}\,. \tag{14.13}$$

14.3 Dispersion Relations & the Q Parameter

14.3.1 Linear stability analysis of stellar systems

In general, the procedure for analyzing the stability of a stellar system is:

1. Start with an equilibrium solution to the CBE and Poisson Equation:

$$f = f_0(\mathbf{r},\mathbf{v})\,, \qquad \Phi = \Phi_0(\mathbf{r})\,. \tag{14.14}$$

2. Introduce perturbations scaled by $\varepsilon \ll 1$:

$$f = f_0(\mathbf{r},\mathbf{v}) + \varepsilon f_1(\mathbf{r},\mathbf{v},t)\,, \qquad \Phi = \Phi_0(\mathbf{r}) + \varepsilon\Phi_1(\mathbf{r},t)\,. \tag{14.15}$$

3. Plug these perturbed functions into the CBE and Poisson Equation, and keep only terms of $O(\varepsilon)$. This yields the *linearized* forms of the CBE

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{r}} - \nabla\Phi_0\cdot\frac{\partial f_1}{\partial \mathbf{v}} - \nabla\Phi_1\cdot\frac{\partial f_0}{\partial \mathbf{v}} = 0\,. \tag{14.16}$$

   and Poisson Equations

$$\nabla^2\Phi_1 = 4\pi G\int d^3\mathbf{v}\, f_1(\mathbf{r},\mathbf{v},t)\,. \tag{14.17}$$

4. Solve the linearized equations to find the time-development of an initial $f_1(\mathbf{r},\mathbf{v},0)$. If *any* initial perturbation can be shown to grow with time, the system is unstable. To prove stability one must, in principle, consider all possible perturbations, and show that *none* lead to growing solutions.

Local analysis

If the equilibrium solution is spatially homogeneous, *or* if the characteristic length-scale of the perturbations is much smaller than the characteristic length-scale of the system (WKB approximation), the imposed perturbations can be Fourier-analyzed in space and time into components of the form

$$f_1(\mathbf{r},\mathbf{v},t) = g(\mathbf{v})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,, \tag{14.18}$$

where $\mathbf{k}$ is the wave-vector and ω is the frequency of the perturbation. If *any* growing solutions of the linearized CBE exist then there must be solutions of the form in (14.18) which also grow, since any solution can be expressed as a sum of these Fourier components. When (14.18) is inserted into the linearized CBE and Poisson Equations, the result is a *dispersion relation* between ω^2 and $\mathbf{k}$. *If $\omega^2 < 0$ for any value of $\mathbf{k}$ then perturbations with that wave-number are unstable* because then $\omega = i\gamma$ for some real γ, and the corresponding Fourier component grows like

$$f_1 \propto e^{-i\omega t} = e^{\gamma t}\,. \tag{14.19}$$

14.3.2 Results of WKB analysis for disks

The WKB analysis of a differentially-rotating disk galaxy is covered in BT87, Ch. 6.2. Here I will only quote results for axisymmetric perturbations, which locally have the form

$$f_1 \propto e^{i(kR-\omega t)}\,; \tag{14.20}$$

it turns out that such perturbations are sufficiently general to expose the most important physical effects. The dispersion relations resulting from such perturbations involve a quantity not yet mentioned: the radial or *epicyclic* period of a star on a nearly circular orbit,

$$\kappa = \sqrt{R\frac{d\Omega^2}{dR} + 4\Omega^2}\,, \tag{14.21}$$

where $\Omega(R)$ is the angular velocity of the circular orbit at radius R.

For a *gas* disk, the dispersion relation is

$$\omega^2 = \kappa^2 - 2\pi G\Sigma|k| + k^2 v_{\mathrm{s}}\,, \tag{14.22}$$

where v_{s} is the speed of sound in the gas.

For a *stellar* disk, the dispersion relation depends on the detailed form of the distribution function. If the random stellar velocities in the disk are assumed to have a gaussian distribution, the dispersion relation is

$$\omega^2 = \kappa^2 - 2\pi G\Sigma|k|\mathscr{F}\left(\frac{\Omega}{\kappa}, \frac{k^2\sigma_R^2}{\kappa^2}\right)\,, \tag{14.23}$$

where σ_R is the radial velocity dispersion and the *reduction factor* $\mathscr{F}(s,\chi)$ is given in Eq. (6-45) of BT87. Note that $\mathscr{F}(s,0) = 1$; thus (14.22) and (14.23) yield identical results in the limiting case where $v_{\mathrm{s}} = \sigma_R = 0$. This is reasonable since the dynamical stability of a perfectly 'cold' disk should not depend on its make-up.

In either case, local stability against axisymmetric perturbations is assured if $\omega^2 > 0$ for all values of k. This condition implies that

$$Q_{\mathrm{gas}} \equiv \frac{\kappa v_{\mathrm{s}}}{\pi G\Sigma} > 1\,, \tag{14.24}$$

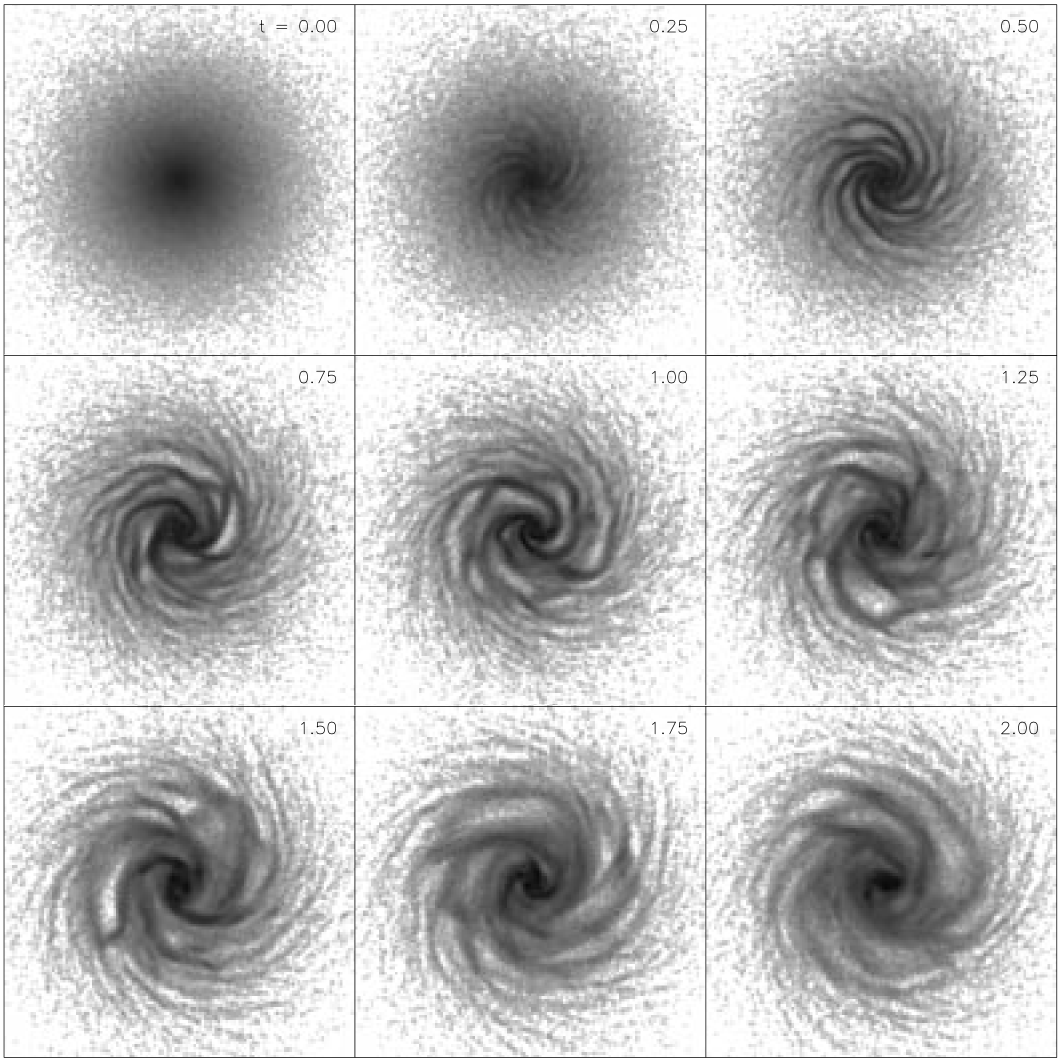

Figure 14.1: Unstable disk simulation. Initially, this disk has $Q \simeq 0.63$, producing the violent, essentially local instability seen here. The galaxy model includes a bulge and a halo (not shown); the disk is 15% of the total mass, and a version with a higher Q value is stable (Fig. 15.2). Each frame is 15 disk scale lengths on a side; times are given in units of the rotation period at ~ 3 disk scale lengths.

for a stable gas disk, and

$$Q_{\rm stars} \equiv \frac{\kappa \sigma_R}{3.36 G \Sigma} > 1 \,, \tag{14.25}$$

for a stable stellar disk (Toomre 1964).

It's worth comparing (14.25), the product of a WKB analysis, with (14.13), which was derived by simple physical considerations. The condition $Q_{\rm stars} > 1$ may be rearranged to give

$$\sigma_R > 3.36 \frac{G\Sigma}{\kappa} \,. \tag{14.26}$$

This is very similar to (14.13); apart from the numerical factor, the only difference is that the epicyclic frequency κ replaces the Oort constant B in the denominator.

Fig. 14.1 shows an N-body simulation of a violently unstable disk. At the start of the simulation, the disk has a $Q \simeq 0.63$; this is small enough, compared to unity, to produce dramatic but highly transient spirals. The galaxy model used for this experiment includes a bulge and a dark halo (not shown); these components help insure global stability (Fig. 15.2), but cannot prevent the instability shown here. After only about 2 rotation periods the instability has "heated" the disk enough to perceptibly reduce the vehemence of the spiral-making.

14.4 Q Parameters of Real Galaxies

An estimate of Q for the solar neighborhood is given in BT87, Ch. 6.2. For the solar neighborhood, the surface density is $\Sigma \simeq 75\,\mathrm{M_\odot\,pc^{-2}}$ and the epicyclic period is $\kappa \simeq 36\,\mathrm{km\,s^{-1}\,kpc^{-1}}$. The radial velocity dispersion, averaged over the vertical extent of the disk, is $\sigma_R \simeq 45\,\mathrm{km\,s^{-1}}$. To account for the finite thickness and gas content of the galactic disk, the coefficient of 3.36 in (14.25) should be reduced to about 2.9 (Toomre 1974). The result is that for the solar neighborhood $Q \simeq 1.7$, so it appears that the Milky Way is locally stable.

For other disk galaxies, the radial dispersion profile may be estimated by comparing gaseous and stellar rotation velocities; the latter lag the former by an amount proportional to σ_R^2 due to asymmetric drift. The few galaxies which have been studied so far yield $Q = 1.5$ to 2; moreover, Q appears to be fairly independent of the radius R (GKvdK89, Ch. 10.2).

It is easy to understand why $Q > 1$; if galactic disks were locally unstable to gravitational collapse then massive clumps of stars would form and scatter other stars, increasing the velocity dispersion until $Q = 1$ was reached. But the actual mechanism(s) responsible for randomizing the velocities of disk stars are not completely understood. Scattering by giant molecular clouds (which may represent gravitationally-collapsed clumps in the *gaseous* disk) can explain part of the velocity increase, but apparently not all of it (*e.g.* Wielen & Fuchs 1990).