Outline

- Introduction: Modern astronomy and the power of quantitative spectroscopy
- Basic assumptions for "classic" stellar atmospheres: geometry, hydrostatic equilibrium, conservation of momentum-mass-energy, LTE (Planck, Maxwell)
- Radiative transfer: definitions, opacity, emissivity, optical depth, exact and approximate solutions, moments of intensity, Lambda operator, diffusion (Eddington) approximation, limb darkening, grey atmosphere, solar models
- **Energy transport:** Radiative equilibrium and convection
- Atomic radiation processes: Einstein coefficients, line broadening, continuous processes and scattering (Thomson, Rayleigh)
- Damped Ly-α systems: ISM of the Galaxy, IGM
- Excitation and ionization (Boltzmann, Saha), partition function
- Example: Stellar spectral types
- Non-LTE: basic concept and examples
- 2-level atom, formation of spectral lines, curves of growth
- Recombination theory in stellar envelopes and gaseous nebulae
- Stellar winds: introduction to line transfer with velocity fields and radiation driven winds
4. Transport of energy: convection

- convection in stars, solar granulation
- Schwarzschild criterion for convective instability
- mixing-length theory
Convection as a means of energy transport

Mechanisms of energy transport

a. radiation: \( F_{\text{rad}} \) (most important)

b. convection: \( F_{\text{conv}} \) (important especially in cool stars)

c. heat production: e.g. in the transition between solar cromosphere and corona

d. radial flow of matter: corona and stellar wind

e. sound waves: cromosphere and corona

The sum \( F(r) = F_{\text{rad}}(r) + F_{\text{conv}}(r) \) must be used to satisfy the condition of energy equilibrium, i.e. no sources and sinks of energy in the atmosphere

\[
4\pi r^2 F(r) = \text{const} = L
\]
Solar granulation

\[ \Delta T \approx 100 \text{ K} \quad \text{v} \approx \text{few km/s} \]

resolved size $\sim 500$ km (0.7 arcsec)
Solar granulation

warm gas is pulled to the surface
cool dense gas falls back into the deeper atmosphere
Convection as a means of energy transport

Hot stars (type OBA)
Radiative transport more efficient in atmospheres
**CONVECTIVE CORES**

Cool stars (F and later)
Convective transport more important in atmospheres (dominant in coolest stars)
**OUTER CONVECTIVE ZONE**

**When does convection occur? → strategy**

Start with an atmosphere in radiative equilibrium.

Displace a mass element via small perturbation. If such displacement grows larger and larger through buoyancy forces → instability (convection)
The Schwarzschild criterion for instability

Assume a mass element in the photosphere slowly \( v < v_{\text{sound}} \) moves upwards (perturbation) adiabatically (no energy exchanged)

Ambient density and pressure decreases \( \rho \rightarrow \rho_a \)

mass element (‘cell’) adjusts and expands, changes \( \rho, T \) inside

2 cases:

a. \( \rho_i > \rho_a \): the cell falls back \( \rightarrow \) stable
b. \( \rho_i < \rho_a \): cell rises \( \rightarrow \) unstable
The Schwarzschild criterion for instability

at the end of the adiabatic expansion $\rho$ and $T$ of the cell: $\rho \rightarrow \rho_i, T \rightarrow T_i$

denoting with subscript ‘ad’ the adiabatic quantities and with ‘rad’ the radiative ones
we can substitute $\rho_i \rightarrow \rho_{ad}, T_i \rightarrow T_{ad}$ and $\rho_a \rightarrow \rho_{rad}, T_a \rightarrow T_{rad}$

the change in density over the radial distance $\Delta x$: $\Delta \rho_{ad} = \frac{d\rho}{dx}_{ad} \Delta x$

we obtain stability if: $|\Delta \rho_{ad}| < |\Delta \rho_{rad}|$

outwards: $\rho$ decreases! $\rightarrow$ density of cell is larger than surroundings,
inwards: $\rho$ increases! $\rightarrow$ density of cell smaller than surroundings

2 cases:

a. $\rho_{ad} > \rho_{rad}$: the cell falls back $\rightarrow$ stable

b. $\rho_{ad} < \rho_{rad}$: cell rises $\rightarrow$ unstable

[2 cases: a. $\rho_{ad} > \rho_{rad}$: the cell falls back $\rightarrow$ stable

b. $\rho_{ad} < \rho_{rad}$: cell rises $\rightarrow$ unstable]
The Schwarzschild criterion for instability

We assume that the motion is slow, i.e. with \( v << v_{\text{sound}} \)

\( \rightarrow \) pressure equilibrium !!!

\( \rightarrow \) pressure inside cell equal to outside: \( P_{\text{rad}} = P_{\text{ad}} \)

from the equation of state: \( P \sim \rho \, T \rightarrow \rho_{\text{rad}} \, T_{\text{rad}} = \rho_{\text{ad}} \, T_{\text{ad}} \)

\( \rightarrow \) equivalently stability if:

\[
\left| \frac{dT}{dx} \right|_{\text{ad}} > \left| \frac{dT}{dx} \right|_{\text{rad}}
\]

equivalent to old criterium

\[
\left| \frac{d\rho}{dx} \right|_{\text{ad}} < \left| \frac{d\rho}{dx} \right|_{\text{rad}}
\]
The Schwarzschild criterion for instability

**Equation of hydrostatic equilibrium:**

\[ dP_{\text{gas}} = -g \, \rho(x) \, dx \]

\[ P_{\text{gas}} = (k/m_H \, \mu) \, \rho \, T \]

\[
\left| \frac{dT}{dP} \right| = \left| \frac{dT}{dP} \frac{dP}{dx} \right| = \left| \frac{dT}{dP} g \rho \right| = \left| \frac{dT}{dP} g \frac{m_H \mu}{kT} P \right| = g \frac{m_H \mu}{k} \left| \frac{d \ln T}{d \ln P} \right|
\]

→ **Stability** if:

\[
\left| \frac{d \ln T}{d \ln P} \right|_{\text{ad}} > \left| \frac{d \ln T}{d \ln P} \right|_{\text{rad}}
\]

**Schwarzschild criterion**

**Instability:** when the temperature gradient in the stellar atmosphere is larger than the adiabatic gradient → convection !!!
The Schwarzschild criterion for instability

to evaluate \( \frac{d \ln T}{d \ln P} \)_{ad}

Adiabatic process: \( P \sim \rho^\gamma \)

\( \gamma = C_p/C_v \)

perfect monoatomic gas which is completely ionized or completely neutral: \( \gamma = 5/3 \)

from the equation of state: \( P = \rho k T/\mu \) \( \rightarrow \) \( P^{\gamma^{-1}} \sim T^\gamma \)
The Schwarzschild criterion for instability

\[
\left. \frac{d \ln T}{d \ln P} \right|_{ad} = \frac{\gamma - 1}{\gamma} = 1 - \frac{1}{\gamma}
\]

“simple” plasma with constant ionization $\rightarrow \gamma = 5/3 \rightarrow \left. \frac{d \ln T}{d \ln P} \right|_{ad} = 0.4$

changing degree of ionization $\rightarrow$ plasma undergoes phase transition

$\rightarrow$ specific heat capacities $dQ = C \, dT$ not constant!!$

$\rightarrow$ adiabatic exponent $\gamma$ changes

\[
\text{H} \quad \leftrightarrow \quad p + e^- \quad \gamma \neq 5/3
\]
The Schwarzschild criterion for instability

the general case:

\[ \left| \frac{d \ln T}{d \ln P} \right|_{ad} = \frac{2 + X_{\text{ion}} (1 - X_{\text{ion}}) \left( \frac{5}{2} + \frac{E_{\text{ion}}}{kT} \right)}{5 + X_{\text{ion}} (1 - X_{\text{ion}}) \left( \frac{5}{2} + \frac{E_{\text{ion}}}{kT} \right)^2} \]

\[ X_{\text{ion}} = \frac{n_e}{n_p + n(H)} \quad \text{degree of ionization} \]

\[ E_{\text{ion}} \quad \text{ionization energy} \]

for \( X_{\text{ion}} = 0 \) or \( 1 \):

\[ \left| \frac{d \ln T}{d \ln P} \right|_{ad} = 0.4 \]

for \( X_{\text{ion}} = 1/2 \) a minimum is reached

\[ \left| \frac{d \ln T}{d \ln P} \right|_{ad} = 0.07 \]

(example: inner boundary of solar photosphere at \( T = 9,000 \) K)

Isocontours of adiabatic gradient in (logP, logT)-plane

note that gradient is small, where H, HeI, HeII change ionization

Unsoeld, 68
Conditions for convection

stability when

\[ \frac{d \ln T}{d \ln P}_{ad} < \frac{d \ln T}{d \ln P}_{rad} \]

can depend on:

1. small
2. large

1. HOT STARS: \( T_{\text{eff}} > 10,000 \text{ K} \)
   H fully ionized \( \rightarrow \gamma = 5/3 \)
   \[ \frac{d \ln T}{d \ln P}_{ad} = 0.4 \]
   convection NOT important

1. COOL STARS: \( T_{\text{eff}} < 10,000 \text{ K} \)
   contain zones with \( X_{\text{ion}} = 0.5 \)
   \[ \frac{d \ln T}{d \ln P}_{ad} = 0.07 \]
   convective zones

\[ \gamma-\text{effect} \]
Conditions for convection

2. \[ T^4 = \frac{3}{4} T_{\text{eff}}^4 (\bar{\tau} + \frac{2}{3}) \implies \left| \frac{dT}{dx} \right|_{\text{rad}} = \frac{\bar{\kappa}}{4 T^3} \frac{3}{4} T_{\text{eff}}^4 \]

\[
\left| \frac{d \ln T}{d \ln P} \right|_{\text{rad}} = \frac{k}{g \mu m_H} \frac{\bar{\kappa}}{4 T^3} \frac{3}{4} T_{\text{eff}}^4 = \frac{3}{16} \left( \frac{T_{\text{eff}}}{T} \right)^4 \bar{\kappa} H \approx \frac{3}{16} \frac{\pi P}{\sigma g \rho T^4 \bar{\kappa} F}
\]

large (convective instability) when \( \kappa \) large
(less important: flux \( F \) or scale height \( H \) large)

Example: in the Sun this occurs around \( T \approx 9,000 \) K
because of strong Balmer absorption from H 2nd level

\( \kappa \)-effect

\( \gamma \)-effect

\( \bar{\kappa} \)-effect

hydrogen convection zone in the sun
Conditions for convection

- The previous slides considered only the simple case of atomic hydrogen ionization and hydrogen opacities. This is good enough to explain the hydrogen convection zone of the sun.

- However, for stars substantially cooler than the sun molecular phase transitions and molecular opacities become important as well. This is why convection becomes more and more important for all cool stars in their entire atmospheres. The effects of scale heights also contribute for red giants and supergiants.

- Even for hot stars, helium and metal ionization can lead to convection (see Groth, Kudritzki & Heber, 1984, A&A 152, 107). However, here convection is only important to prevent gravitational settling of elements, the convective flux is small compared with the radiative flux and radiative equilibrium is still a valid condition.
simple approach to a complicated phenomenon:

a. suppose atmosphere becomes unstable at \( r = r_0 \to \) mass element rises for a characteristic distance \( L \) (mixing length) to \( r_0 + L \)

b. the cell excess energy is released into the ambient medium

c. cell cools, sinks back down, absorbs energy, rises again, ...

In this process the temperature gradient becomes shallower than in the purely radiative case

given the pressure scale height (see ch. 2)

\[
H = \frac{kT}{g m_H \mu}
\]

we parameterize the mixing length by

(adjustable parameter)

\[
\alpha = \frac{L}{H} \quad \alpha = 0.5 - 1.5
\]
Mixing length theory: further assumptions

simple approach to a complicated phenomenon:

d. mixing length $L$ equal for all cells

e. the velocity $v$ of all cells is equal

Note: assumptions d., e. are made ad hoc for simplicity.

There is no real justification for them
Mixing length theory

\[ \Delta T = \left[ \left| \frac{dT}{dr} \right|_{\text{rad}} - \left| \frac{dT}{dr} \right|_{\text{ad}} \right] \Delta r \]

> 0 under the conditions for convection

\[ \Delta T^{\text{max}} = \left[ \left| \frac{dT}{dr} \right|_{\text{rad}} - \left| \frac{dT}{dr} \right|_{\text{ad}} \right] l \]
Mixing length theory: energy flux

for a cell travelling with speed $v$ the flux of energy is:

\[ \text{Flux} = \text{mass flow} \times \text{heat energy per gram} \]

\[ \pi F_{\text{conv}} = \rho v \times dQ = \rho v C_p \Delta T \]

**Estimate of $v$:**

1. **buoyancy force:** $|f_b| = g |\Delta \rho|$  
   $\Delta \rho$: density difference cell – surroundings. We want to express it in terms of $\Delta T$, which is known $\Rightarrow$

2. **equation of state:** $P = \frac{\rho k T}{\mu m_H} \Rightarrow \frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T} - \frac{d\mu}{\mu}$
Mixing length theory: energy flux

in pressure equilibrium:

\[
\frac{dP}{P} = 0 = \frac{d\rho}{\rho} + \frac{dT}{T} - \frac{d\mu}{\mu}
\]

\[
d\rho = -\rho \left( \frac{dT}{T} - \frac{d\mu}{\mu} \right) = -\rho \frac{dT}{T} \left( 1 - \frac{d\ln \mu}{d\ln T} \right)
\]

\[
|\Delta \rho| = \frac{\rho}{T} \Delta T \left| 1 - \frac{d\ln \mu}{d\ln T} \right|
\]

\[
\Delta T = \left[ \left| \frac{dT}{dr} \right|_{\text{rad}} - \left| \frac{dT}{dr} \right|_{\text{ad}} \right] \Delta r
\]

3. work done by buoyancy force:

\[
w = \int_{0}^{l} |f_b| d(\Delta r) = \int_{0}^{l} g |\Delta \rho| d(\Delta r)
\]

crude approximation: we assume integrand to be constant over Integration interval
Mixing length theory: energy flux

\[ w = g \frac{\rho}{T} \left| 1 - \frac{d \ln \mu}{d \ln T} \right| \left[ \left| \frac{dT}{dr} \right|_{\text{rad}} - \left| \frac{dT}{dr} \right|_{\text{ad}} \right] \times \frac{1}{2} l^2 \]

4. equate kinetic energy \( \rho v^2 / 2 \) to work

\[ v = \left[ g \frac{\rho}{T} \left| 1 - \frac{d \ln \mu}{d \ln T} \right| \right]^{1/2} \left[ \left| \frac{dT}{dr} \right|_{\text{rad}} - \left| \frac{dT}{dr} \right|_{\text{ad}} \right]^{1/2} l \]

\[ \pi F_{\text{conv}} = \rho vC_p \Delta T = \rho C_p \left[ g \frac{\rho}{T} \left| 1 - \frac{d \ln \mu}{d \ln T} \right| \right]^{1/2} \left[ \left| \frac{dT}{dr} \right|_{\text{rad}} - \left| \frac{dT}{dr} \right|_{\text{ad}} \right]^{3/2} l^2 \]
Mixing length theory: energy flux

re-arranging the equation of state:

\[
\frac{dT}{dr} = \frac{g \mu m_H}{k} \frac{d\ln T}{d\ln P} = \frac{T}{H} \frac{d\ln T}{d\ln P}
\]

\[
\pi F_{\text{conv}} = \rho C_p \alpha^2 T \left[ g H \left( 1 - \frac{d\ln \mu}{d\ln T} \right) \right]^{1/2} \left[ \frac{d\ln T}{d\ln P} \right]_{\text{rad}} - \left[ \frac{d\ln T}{d\ln P} \right]_{\text{ad}}^{3/2}
\]

Note that \( F_{\text{conv}} \sim \alpha^2 \), \( \alpha \) is a free parameter !!!!!!!

Note that \( F_{\text{conv}} \sim T^{1.5} \), whereas \( F_{\text{rad}} \sim T_{\text{eff}}^4 \)

\( \rightarrow \) convection not effective for hot stars
Mixing length theory: energy flux

We finally require that the total energy flux

\[ \pi F = \pi F_{\text{rad}} + \pi F_{\text{conv}} = \sigma T_{\text{eff}}^4 \]

Since the T stratification is first done on the assumption:

\[ \pi F_{\text{rad}} = \sigma T_{\text{eff}}^4 \]

a correction \( \Delta T(\tau) \) must be applied iteratively to calculate the correct T stratification if the instability criterion for convection is found to be satisfied.
Mixing length theory: energy flux

comparison of numerical models with mixing-length theory results for solar convection

Abbett et al. 1997
3D Hydrodynamical modeling

Mixing length theory is simple and allows an analytical treatment

More recent studies of stellar convection make use of radiative–hydrodynamical numerical simulations in 3D (account for time dependence), including radiation transfer. Need a lot of CPU power.

Nordlund 1999

Sun (L71D09), $T_{\text{eff}}=5770$ K, logg=4.44
212 x 106 grid points, 11540 s ($\Delta t=20$ s)

Matthias Steffen, Bernd Freitag

Time: 18880.0 sec

Nordlund 1999

Freitag & Steffen
3D Hydrodynamical modeling

α Orionis (Freitag 2000)