2. Basic assumptions for stellar atmospheres

1. geometry, stationarity
2. conservation of momentum, mass
3. conservation of energy
4. Local Thermodynamic Equilibrium
1. Geometry

Stars as gaseous spheres → **spherical symmetry**

Exceptions: rapidly rotating stars

- Be stars $v_{\text{rot}} = 300 – 400 \text{ km/s}$
  - (Sun $v_{\text{rot}} = 2 \text{ km/s}$)

For stellar photospheres typically: $\Delta r/R \ll 1$

- Sun: $R_o = 700,000 \text{ km}$
  - photosphere $\Delta r = 300 \text{ km}$: $\Delta r/R = 4 \times 10^{-4}$
  - chromosphere $\Delta r = 3000 \text{ km}$: $\Delta r/R = 4 \times 10^{-3}$
  - corona: $\Delta r/R \sim 3$
As long as $\Delta r/R << 1$ : plane-parallel symmetry

consider a light ray through the atmosphere:

$\Delta r/R << 1$

$\Delta r/R \sim 1$

lines of constant temperature and density

Path of light

**Plane-parallel symmetry**
very small curvature ($\alpha \sim \beta$)
Solar photo/cromosphere, dwarfs, (giants)

**Spherical symmetry**
significant curvature ($\alpha \neq \beta$)
Solar corona, supergiants, expanding envelopes (OBA stars), novae, SN
Homogeneity & stationarity

We assume the atmosphere to be homogeneous

Counter-examples:
sunspots, granulation, non-radial pulsations
clumps and shocks in hot star winds
magnetic Ap stars

and stationary
Most spectra are time-independent: \( \partial/\partial t = 0 \) (if we don’t look carefully enough…)

Exceptions: explosive phenomena (SN), stellar pulsations, magnetic stars, mass transfer in close binaries
2. Conservation of momentum & mass

Consider a mass element $dm$ in a spherically symmetric atmosphere.

The acceleration of the mass element results from the sum of all forces acting on $dm$, according to Newton’s 2nd law:

$$ dm \frac{d}{dt} v(r,t) = \sum_i df_i = F $$
a. Hydrostatic equilibrium

Assuming a hydrostatic stratification: $v(r,t) = 0$

$$0 = \sum_i df_i$$

**Gravitational forces:**
$$df_{grav} = -G \frac{M_r \cdot dm}{r^2} = -g(r) \, dm$$

**Gas Pressure forces:**
$$df_{p,gas} = -A \, P(r + dr) + A \, P(r) = -A \frac{dP}{dr} \, dr$$

**Radiation forces:**
$$df_{rad} = g_{rad}(r) \, dm = \frac{\text{absorbed photon momentum}}{dt}$$
In equilibrium:

\[ 0 = \sum_i df_i = -g(r)dm - A \frac{dP}{dr}dr + g_{rad}dm \]

and substituting \( dm = A \rho dr \):

\[ -g(r)\rho Adr - A \frac{dP}{dr}dr + g_{rad}\rho Adr = 0 \]

\[ \frac{dP}{dr} = -\rho(r)[g(r) - g_{rad}] \]

Hydrostatic equilibrium in spherical symmetry
Approximation for \( g(r) \)

The mass within the atmosphere \( M(r) \) – \( M(R) \) \( \ll \) \( M(R) = M^* \)

\[ g(r) = \frac{G M^*}{r^2} \]

Example: take a geometrically thin photosphere

\[
\Delta M_{\text{phot}} \approx \rho \frac{4\pi}{3} \left[ (R + \Delta r)^3 - R^3 \right] = \rho \frac{4\pi}{3} \left[ R^3 \left(1 + \frac{\Delta r}{R}\right)^3 - R^3 \right]
\]

\[
\approx \rho \frac{4\pi}{3} R^3 \left[ 1 + 3 \frac{\Delta r}{R} - 1 \right] = \rho 4\pi R^2 \Delta r
\]
Example: the sun

\[ R = 7 \times 10^{10} \text{ cm}, \ \Delta r = 3 \times 10^7 \text{ cm}, \ \rho \sim m_N, \ N = 1.7 \times 10^{-24} \times 10^{15} \text{ g/cm}^3: \]

\[ \Delta M_{\text{phot}} = 3 \times 10^{21} \text{ g} \quad \Delta M_{\text{phot}}/M = 10^{-12} \]

Moreover in plane-parallel symmetry:

\[ \Delta r/R \ll 1 \]

\[ g(r) = \text{const} \]

\[ g = G M_\ast/R^2 \]

<table>
<thead>
<tr>
<th>Type</th>
<th>log g</th>
</tr>
</thead>
<tbody>
<tr>
<td>Main sequence star</td>
<td>4 (cgs: cm/s²)</td>
</tr>
<tr>
<td>Supergiant</td>
<td>3.5 – 0.8</td>
</tr>
<tr>
<td>White dwarf</td>
<td>8</td>
</tr>
<tr>
<td>Sun</td>
<td>4.44</td>
</tr>
<tr>
<td>Earth</td>
<td>3.0</td>
</tr>
</tbody>
</table>
Barometric formula

Assume:

• plane-parallel symmetry
• \( g_{\text{rad}} \ll g \)
• ideal gas:

\[
P = \frac{\rho kT}{m_H \mu} = \frac{\rho RT}{\mu} = N_g kT
\]

• and:

\[
\frac{1}{\rho} \frac{d\rho}{dr} \gg \frac{1}{T} \frac{dT}{dr}
\]

Note:

\[
\frac{kT}{m_H \mu} = v_{\text{sound}}^2
\]

\( k \): Boltzmann constant = \( 1.38 \times 10^{-16} \) erg K\(^{-1} \)

\( m_H \): mass of H atom = \( 1.66 \times 10^{-24} \) g

\( \mu \): mean molecular weight per free particle = \( \langle m \rangle / m_H \)

\( R = k N_A = 8.314 \times 10^7 \) erg/mole/K

\[
\frac{dP}{dr} = -\rho(r)g(r)
\]
Barometric formula

\[ \frac{dP}{dr} = -\rho(r)g(r) \]

\[ \frac{k}{m_H \mu} \left[ T \frac{d\rho}{dr} + \rho \frac{dT}{dr} \right] = -g \rho \]

equation is negligible by assumption

\[ \frac{1}{\rho} \frac{d\rho}{dr} = -\frac{g m_H \mu}{kT} \]

and:

\[ H = \frac{kT}{g m_H \mu} \]

pressure scale height

H(Sun) = 150 km

solution:

\[ \rho(r) = \rho(r_0) e^{-\frac{r-r_0}{H}} \]
Approximate solution:
Barometric formula

\[ \frac{dP}{dr} = -\rho(r)g(r) \]

\[ \rho(r) = \rho(r_0) e^{-\frac{r-r_0}{H}} \]

- exponential decline of density
- scale length \( H \sim T/g \)
- explains why atmospheres are so thin
b. radial flow

Let us now consider the case in which there are deviations from hydrostatic equilibrium and \( v(r,t) \neq 0 \).

Newton’s law:

\[
dm \frac{d}{dt} v(r,t) = \sum_i df_i
\]

\[
\frac{d}{dt} v(r,t) = \lim_{\Delta t \to 0} \frac{v(r + \Delta r, t + \Delta t) - v(r,t)}{\Delta t}
\]

Taylor expansion:

\[
v(r + \Delta r, t + \Delta t) = v(r,t) + \frac{\partial v}{\partial r} \Delta r + \frac{\partial v}{\partial t} \Delta t
\]

and

\[
\Delta r = v \Delta t
\]

In general:

\[
\frac{d}{dt} \alpha(r,t) = \frac{\partial}{\partial t} \alpha(r,t) + v(r,t) \times \text{grad} \{\alpha(r,t)\}
\]
For the assumption of stationarity $\partial l/\partial t = 0$

\[
\frac{d}{dt} v(r,t) = v \frac{\partial v}{\partial r}
\]

\[
dm \frac{d}{dt} v(r,t) = dm v \frac{\partial v}{\partial r} = \sum_i df_i
\]

\[
dm = A \rho dr
\]

\[
-g(r)dm - A \frac{dP}{dr} dr + g_{rad} dm
\]

\[
\frac{dP}{dr} = -\rho [g(r) - g_{rad} + v \frac{dv}{dr}]
\]
equation of continuity, mass conservation

\[ \dot{M} = \frac{d}{dt} M = 4\pi r^2 \rho v \]

From: \[ \frac{dP}{dr} = -\rho [g(r) - g_{rad} + v \frac{dv}{dr}] \]

and from the equation of state:

\[ \frac{dP}{dr} = \frac{kT}{m_H \mu} \]
\[ \frac{d\rho}{dr} = v_{\text{sound}} \frac{d\rho}{dr} \]

\[ \frac{dv}{dr} = -\frac{2\dot{M}}{4\pi r^3 \rho} - \frac{\dot{M}}{4\pi r^2 \rho^2} \frac{d\rho}{dr} = -\frac{2v}{r} - \frac{v}{\rho} \frac{d\rho}{dr} \]

\[ \rho v \frac{dv}{dr} = -2\rho \frac{v^2}{r} - \frac{v^2}{r} \frac{d\rho}{dr} \]
Hydrodynamic equation of motion

\[
(v_{\text{sound}}^2 - v^2) \frac{d\rho}{dr} = -\rho [g(r) - g_{\text{rad}} - 2 \frac{v^2}{r}] 
\]

with

\[
\text{escape velocity: } \frac{mv_{\text{esc}}^2(r)}{2} = \frac{GmM}{r} \Rightarrow mrg \rightarrow v_{\text{esc}}^2(r) = 2gr 
\]

\[
(v_{\text{sound}}^2 - v^2) \frac{d\rho}{dr} = -\rho g(r) [1 - \frac{g_{\text{rad}}}{g(r)} - 4 \frac{v^2}{v_{\text{esc}}^2(r)}] 
\]

When \( v \ll v_{\text{sound}} \): practically hydrostatic solution (\( v = 0 \)) for density stratification. This is reached well below the sonic point (where \( v = v_{\text{sound}} \)).

example: \( v_{\text{sound}} = 6 \text{ km/s for solar photosphere, 20 km/s for O stars} \)

\( v_{\text{esc}} = 100 \text{ to 1000 km/s for main sequence and supergiant stars} \)

\[
(v_{\text{sound}}/v_{\text{esc}})^2 \ll 1 
\]
3. Conservation of energy

Stellar interior: production of energy via nuclear reactions

Stellar atmosphere: negligible production of energy

the energy flux is conserved at any given radius

\[ F(r) = \frac{\text{energy}}{\text{area}\cdot\text{time}} \]

\[ 4\pi r^2 F(r) = \text{const} = \text{luminosity} \ L \]

In spherical symmetry: \( r^2 F(r) = \text{const} \)

\[ F(r) \sim \frac{1}{r^2} \]

Plane-parallel: \( r^2 \sim R^2 \sim \text{const} \)

\[ F(r) \sim \text{const} \]
4. Concepts of Thermodynamics

Radiation field

Consider a closed “cavity” in thermodynamic equilibrium TE (photons and particles in equilibrium at some temperature T).

The specific intensity emitted (through a small hole) is (energy per area, per unit time, per unit frequency, per unit solid angle) (Planck function):

\[
I_\nu \, d\nu = B_\nu (T) \, d\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} d\nu \quad \text{(erg cm}^{-2} \text{s}^{-1} \text{Hz}^{-1} \text{sr}^{-1})}
\]

**black body radiation:** universal function dependent only on T not on chemical composition, direction, place, etc.

Note: stars don’t radiate as blackbodies, since they are not closed systems !!!
But their radiation follows the Planck function qualitatively
properties of Planck function

a. $B_\nu(T_1) > B_\nu(T_2)$  (monotonic) for every $T_1 > T_2$ (no function crossing)

b. $\nu_{\text{max}}/T = \text{const}$

c. Stefan-Boltzmann law: (total flux)

\[ F = \pi \int_0^\infty B_\nu(T) \, d\nu = \sigma T^4 \]

$\sigma = 5.67 \times 10^{-5} \text{ erg cm}^{-2} \text{s}^{-1} \text{K}^{-4}$

\[ \lambda_{\text{max}} = \frac{2.898 \times 10^7}{T} \text{ Å} \text{ for } B_\lambda \]

\[ \lambda_{\text{max}} = \frac{5.099 \times 10^7}{T} \text{ Å} \text{ for } B_\nu \]

\[ h\nu/kT >> 1: \text{Wien} \]

\[ B_\nu(T) \approx \frac{2h\nu^3}{c^2} e^{-h\nu/kT} \]

\[ h\nu/kT << 1: \text{Rayleigh-Jeans} \]

\[ B_\nu(T) \approx \frac{2h\nu^2}{c^2} kT \]

Rybicki & Lightman, 79
Concepts of Thermodynamics

Gas particles

1. Velocity distribution

In complete equilibrium this is given by Maxwell distribution (needed, e.g., for collisional rates):

\[ f(v)dv = 4\pi v^2 \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{mv^2}{2kT}} dv \]

most probable speed: \( v_p = \left(\frac{2kT}{m}\right)^{1/2} \)

average speed: \( \langle v \rangle = \left(\frac{8kT}{\pi m}\right)^{1/2} \)

r.m.s. speed: \( v_{rms} = \left(3kT/m\right)^{1/2} \)
2. Energy level distribution

Boltzmann’s equation for the population density of excited states in TE

\[ \frac{n_u}{n_l} = \frac{g_u}{g_l} \exp \left( \frac{(E_u - E_l)}{kT} \right) \]

\( g_u, g_l: \) statistical weights = 2J+1: number of degenerate states

= 2n^2 for Hydrogen

\[ \frac{n_n}{n_{tot}} = \frac{g_n}{u} \exp \left( \frac{-E_n}{kT} \right) \]

\[ u = \sum_{i} g_i \exp \left( \frac{-E_i}{kT} \right) \] partition function
3. Kirchhoff’s law

For a thermal emitter in TE

\[ \frac{\varepsilon_v}{\kappa_v} = \frac{\text{emission coefficient}}{\text{absorption coefficient}} = B_v(T) \]

as a consequence of Boltzmann’s equation for excitation and Maxwell’s velocity law.
Local Thermodynamic Equilibrium

A star as a whole (or a stellar atmosphere) is far from being in thermodynamic equilibrium: energy is transported from the center to the surface, driven by a temperature gradient.

But for sufficiently small volume elements $dV$, we can assume TE to hold at a certain temperature $T(r)$.

**Local Thermodynamic Equilibrium (LTE)**

- **good approx**: stellar interior (high density, small distance travelled by photons, nearly-isotropic, thermalized radiation field)
- **bad approx**: gaseous nebulae (low density, non-local radiation field, optically thin)
- stellar atmospheres: optically thick but moderate density
Local Thermodynamic Equilibrium

1. Energy distribution of the gas
determined by local temperature \( T(r) \)
   appearing in Maxwell’s/ Boltzmann’s equations, and Kirchhoff’s law

2. Energy distribution of the photons
photons are carriers of non-local energy through the atmosphere
\[ I_\nu(r) \neq B_\nu[T(r)] \]
   \( I_\nu \) is a superposition of Planck functions originating at different depths in the atmosphere \( \rightarrow \) radiation transfer
LTE vs NLTE

**LTE**

- each volume element separately in thermodynamic equilibrium at temperature $T(r)$

1. $f(v) \, dv = \text{Maxwellian with } T = T(r)$
2. Saha: \( \left( \frac{n_p}{n_e} \right)n_1 / T^{3/2} \exp(-h \nu_1/kT) \)
3. Boltzmann: \( n_i / n_1 = g_i / g_1 \exp(-h \nu_{1i}/kT) \)

However:

- volume elements not closed systems, interactions by photons
- \( \Rightarrow \) LTE non-valid if absorption of photons disrupts equilibrium
LTE vs NLTE

NLTE if
rate of photon absorptions $\gg$ rate of electron collisions

$I_v (T) \gg T^\alpha, \alpha > 1 \quad \Rightarrow \quad n_e T^{1/2}$

LTE

valid: low temperatures & high densities
non-valid: high temperatures & low densities
LTE vs NLTE in hot stars

Kudritzki 1978