VII. Hydrodynamic theory of stellar winds

Observations: winds exist everywhere in the HRD

Hydrodynamic theory needed to describe stellar atmospheres with winds

Unified Model Atmospheres:
- based on the hydrodynamics of radiation driven winds
- fully self-consistent hydrodynamic transition from hydrostatic photosphere to supersonic winds
1. Hydrodynamical equations of an ideal compressible fluid

Definitions:

- $\rho \vec{v}$: momentum density (momentum per unit volume)
- $\vec{f}$: force per unit volume, $\vec{f} dV$ acts on $dV$
- $\Pi$: momentum flux tensor
- $\Pi_{ij}$: flux of momentum of component $i$ perpendicular to direction $j$

$$\Pi_{ij} = \rho v_i v_j + p \delta_{ij}$$

- **Stochastic motion**
- $p = $ gas pressure

**Eq. of continuity**

$$\frac{\partial}{\partial t} \int_V \rho dV = - \int_S \rho \vec{v} d\vec{S}$$

- Gauss theorem

$$\int_S \vec{a} d\vec{S} = \int_V \nabla \vec{a} dV$$

$$\frac{\partial \rho}{\partial t} + \nabla (\rho \vec{v}) = 0$$

Eq. 1
momentum equation

\[ \frac{\partial}{\partial t} \int_V \rho \vec{v} dV = - \oint \Pi d\vec{S} + \int_V \vec{f} dv \]

- change of volume
- momentum with time
- momentum flow through borders of V
- momentum change through external forces

Gauss theorem

\[ \frac{\partial}{\partial t} (\rho \vec{v}) = \nabla \Pi + \vec{f} \]

- definition \( \Pi \)

\[ \frac{\partial}{\partial t} (\rho \vec{v}) + \nabla (\rho \vec{v} \cdot \vec{v}) = - \nabla p + \vec{f} \]

Eq. 2

\[ \rho \frac{D\vec{v}}{Dt} = \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = - \nabla p + \vec{f} \]

note: for any scalar or vector

\[ \frac{\partial (\rho a)}{\partial t} + \nabla (\rho a \vec{v}) = a \{ \frac{\partial \rho}{\partial t} + \nabla (\rho \vec{v}) \} + \rho \{ \frac{\partial a}{\partial t} + (\vec{v} \cdot \nabla) a \} = \rho \frac{Da}{Dt} \]

Eq. **

substantial derivative in co-moving frame

\[ = 0 \quad \text{see, Eq. 1} \]
energy equation

internal energy per unit mass for an ideal gas:

\[ e = \frac{p}{\rho \gamma - 1} = \frac{3}{2} \frac{k}{\mu m_p} T \]

1st law of thermodynamics for volume element \( dV \) with mass \( dm = \rho dV \):

\[ de + pd(\frac{1}{\rho}) = dq \]

work done per unit mass

external energy added or lost per unit mass

energy equation in co-moving frame

\[ \rho \left\{ \frac{De}{Dt} + p \frac{D(\frac{1}{\rho})}{Dt} \right\} = \rho \frac{Dq}{Dt} \]

Eq. 3a

\[ \dot{Q} = \rho \frac{Dq}{Dt} \]

Eq. 3b

external energy added or lost i.e., radiation, sound waves, convection, friction, magnetic fields, etc.
2. Stationary, spherically symmetric winds

- **r** is dimensionless radial coordinate in units of stellar radius as in previous chapters.

**Eq. 1** \( \frac{\partial \rho}{\partial t} + \nabla (\rho \vec{v}) = 0 \)

**Eq. 4a, continuity**

\[ \dot{M} = 4\pi R_*^2 r^2 \rho v \]

**Eq. 2** \( \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right) = -\nabla p + \vec{f} \)

**Eq. 4b, momentum**

\[ \rho v \frac{dv}{R_* dr} - \frac{dp}{R_* dr} - \rho \{ g(r) - g_{\text{add}}(r) \} = f \]

- gravity
- radiation, magnetic fields
\[\rho \left\{ \frac{D e}{D t} + p \frac{D (\frac{1}{\rho})}{D t} \right\} = \dot{Q}\]

\[\frac{D}{D t} \left\{ \frac{\partial}{\partial t} + (\vec{v} \cdot \nabla) \right\} = \dot{Q}\]

\[p \frac{d \rho^{-1}}{dr} = \frac{d}{dr} \left( \frac{p}{\rho} \right) - \frac{1}{\rho} \frac{d p}{d r} \quad e = \frac{p}{\rho \gamma - 1}\]

\[\rho \rho \frac{1}{R_*} \left\{ \frac{d p}{d r} \frac{\gamma}{\rho \gamma - 1} + \rho^{-1} \frac{d p}{d r} \right\} = \dot{Q}\]

Eq. 2
\[\frac{v}{R_*} \frac{dp}{dr} = -v^2 \frac{dv}{R_* dr} - v \{g(r) - g_{add}(r)\}\]

Eq. 4c, energy
\[\frac{v}{R_*} \frac{dv}{dr} = \frac{1}{2} \frac{dv^2}{dr}\]

\[\rho \rho \frac{1}{R_*} \frac{d}{d r} \left\{ \frac{1}{2} v^2 + \frac{p}{\rho \gamma - 1} - \frac{GM_*}{R_*} \right\} = \rho v g_{add}(r) + \dot{Q}\]
Eq. 4c → condition for existence of stellar winds

\[
\frac{d}{dr}\left\{ \frac{1}{2}v^2 + \frac{p}{\rho} \frac{\gamma}{\gamma - 1} - \frac{GM_*}{R_*} \frac{1}{r} \right\} = R_* g_{add}(r) + \frac{R_*}{\rho v} \dot{Q}
\]

integration

\[
E(r) = \frac{1}{2}v^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} - \frac{GM_*}{R_*} \frac{1}{r} = E(r_0) + R_* \int_{r_0}^{r} g_{add}(r)dr + R_* \int_{r_0}^{r} \frac{\dot{Q}}{\rho v}dr
\]

select \( r_0 \approx 1 \) → \( E(r_0) = \frac{1}{2}v_0^2 + \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} - \frac{GM_*}{R_*} \approx -\frac{GM_*}{R_*} \)

\[
\frac{GM_*}{R_*} = \frac{v_{esc}^2}{2}, \quad v_{esc} \approx 100...1000 km/s
\]

\[
\frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} = \frac{5}{2} v_{sound}^2, \quad v_{sound} \approx 5...20 km/s
\]

\( v_0 \leq v_{sound} \)
\[
\frac{1}{2}v^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} - \frac{GM_*}{R_*} \frac{1}{r} \approx -\frac{GM_*}{R_*} + R_* \int_{r_0}^{r} g_{add}(r)dr + R_* \int_{r_0}^{r} \frac{\dot{Q}}{\rho v}dr
\]

\[
r \to \infty
\]

\[
\frac{1}{2}v_\infty^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \approx -\frac{GM_*}{R_*} + R_* \int_{r_0}^{\infty} g_{add}(r)dr + R_* \int_{r_0}^{\infty} \frac{\dot{Q}}{\rho v}dr
\]

left side only \(> 0\), if at least one of the integrals \(\gg 0\)

stellar wind can only escape potential well of the star, if

- significant momentum input

or

- significant energy input
3. “Naïve” theory without additional forces

first simple approach: isothermal wind without additional forces driven by gas pressure only

\[ g_{\text{add}}(r) = 0 \]

- example: solar wind

significant \( \dot{Q} \) in corona \( T_{\text{corona}} \approx 10^6 K \)

pressure driven wind

Eq. 4b

\[ v \frac{dv}{dr} = -\frac{1}{\rho} \frac{dp}{dr} - \frac{GM_*}{R_*} \frac{1}{r^2} \]

with

\[ P = v_s^2 \rho \]

\[ \mu = \frac{1}{2} \]

pure ionized H

\[ v_s^2 = \frac{kT}{\mu m_p} \]

\[ \mu = 0.6...0.65 \]

hot solar gas

\[ v_{\text{esc}}^2 = 2\frac{GM_*}{R_*} \]
\[ v \frac{dv}{dr} = - \frac{v_s^2}{\rho} \frac{d\rho}{dr} - \frac{dv_s^2}{dr} - \frac{1}{2} v_{esc}^2 \frac{1}{r^2} \]

starting point to derive equation of motion for \( v(r) \)
next step: replace \( \rho \) using Eq. 4a

\[ \dot{M} = 4\pi R_*^2 r^2 \rho v \]

\[ \rho(r) = \frac{\dot{M}}{4\pi R_*^2} \frac{1}{r^2 v} \]

\[ \frac{d\rho}{dr} = \frac{\dot{M}}{4\pi R_*^2} \left\{ - \frac{2}{r^3 v} - \frac{1}{r^2 v^2} \frac{dv}{dr} \right\} = -\rho \left\{ \frac{2}{r} + \frac{1}{v} \frac{dv}{dr} \right\} \]

Eq. 5, density/velocity relationship

Eq. 5 can be used to replace density in eq. *
Eq. 6, equation of motion

\[
\left(1 - \frac{v_s^2}{v^2}\right)v \frac{dv}{dr} = \frac{1}{r} \left(2v_s^2 - \frac{1}{2}v_{esc}^2 \frac{1}{r}\right) - \frac{dv_s^2}{dr}
\]

differential eq. for \(v(r)\) of pressure driven winds has singularity called the critical point at

\[
v^2(r_s) = v_s^2 \quad \text{Eq. 7a}
\]

left side Eq. 6 zero at critical point \(\rightarrow\) right side must also be zero

\[
T \sim \text{const} \rightarrow v_s \sim \text{const}.
\]

\[
r_s = \frac{v_{esc}^2}{4v_s^2} \quad \text{Eq. 7b}
\]

because \(v = v_s\) at \(r = r_s\), the critical point is also called the sonic point
Example: solar wind of solar corona
photospheric escape velocity of sun
\[ v_{\text{esc}} = 617 \text{ km/s} \quad T_{\text{corona}} = 2 \times 10^6 \text{ K} \implies v_s = 165 \text{ km/s} \]
\[ \implies r_s = 3.5 \quad \text{sonic point in outer corona} \]

Note: if temperature of corona were as cool as photosphere,
\[ T_{\text{corona}} \approx T_{\text{photosphere}} \implies v_s = 8.5 \text{ km/s} \]
\[ \implies r_s = 1320 \quad \text{we would still have a solar wind, but supersonic just at 6 AU} \]

in principle, all stars will have a (weak) wind, but the solar wind is strong because of high temperature in corona
the importance of the critical point
differential eq. 6 has several families of solutions
critical point → disentangle topology of solutions
→ find physically relevant solution

for simplicity, $v_s = \text{const.}$ with

$$v \frac{dv}{dr} = \frac{v_s^2}{2} \frac{d\left(\frac{v_s^2}{2}\right)}{dr}$$

re-write Eq. 6

$$\left\{1 - \frac{v_s^2}{v^2}\right\} \frac{d\left(\frac{v_s^2}{2}\right)}{dr} = \frac{4}{r} \left\{1 - \frac{r_s}{r}\right\}$$

$$\frac{v^2}{v_s^2} - \ln\frac{v^2}{v_s^2} = 4\ln\frac{r}{r_s} + 4\frac{r_s}{r} + C$$

the implicit solution Eq. 8b for $v(r)$ together with the conditions for the critical point Eq. 7a,b can be used to discuss topology of solution families
\[
\frac{v^2}{v_s^2} - \ln \frac{v^2}{v_s^2} = 4 \ln \frac{r}{r_s} + 4 \frac{r_s}{r} + C
\]

Eq. 8b

Case I: \( v(r) \) exists for \( 1 \leq r \leq \infty \) and is transonic, i.e.

\[
r = r_s : \quad v(r_s) = v_s \quad \text{and} \quad \frac{dv^2}{dr} \neq 0
\]

All monotonic solutions 8b with \( C = -3 \)

What is the value of \( \frac{dv^2}{v_s^2} \) at sonic point?

Eq. 8a

\[
\left\{1 - \frac{v^2}{v_s^2}\right\} \frac{dv^2}{v_s^2} \frac{dr}{dr} = \frac{4}{r} \left\{1 - \frac{r_s}{r}\right\}
\]

\[
\frac{dv^2}{v_s^2} \frac{dr}{dr} = \frac{4}{r} \frac{1 - \frac{r_s}{r}}{1 - \frac{v_s^2}{v^2}} \quad \frac{r}{r_s} \rightarrow \frac{0}{0}
\]

L’Hospital:

\[
\frac{d}{dr} \left\{1 - \frac{r_s}{r}\right\} = -\frac{4}{r^2} (1 - \frac{r_s}{r}) + 4 \frac{r_s}{r^3} \rightarrow \frac{4}{r_s} \frac{1}{r_s}
\]

\[
\frac{d}{dr} \left\{1 - \frac{v_s^2}{v^2}\right\} = \frac{v_s^2}{v^4} \frac{dv^2}{v_s^2} \frac{dr}{dr} \rightarrow \frac{0}{0}
\]

At sonic point

\[
\left[\frac{dv^2}{v_s^2} \frac{dr}{dr}\right]^2 = \frac{4}{r_s^2}
\]

Eq. 9a

Two solutions possible
case I, 1

\[ \frac{d\frac{v^2}{v_s^2}}{dr} = \frac{2}{r_s} = \frac{8v_s^2}{v_{esc}^2} \]

transonic, monotonic, increasing

this solution is realized in solar wind

case I, 2

\[ \frac{d\frac{v^2}{v_s^2}}{dr} = -\frac{2}{r_s} = -\frac{8v_s^2}{v_{esc}^2} \]

transonic, monotonic, decreasing

this solution is not realized

sketch of 2 solutions

\[
\frac{v^2}{v_s^2} - \ln \frac{v^2}{v_s^2} = 4\ln \frac{r}{r_s} + 4\frac{r_s}{r} - 3
\]

I, 1: \( v^2 \geq v_s^2 \) for \( r \geq r_s \)

\[ \frac{v}{v_s} \approx 2\ln \frac{r}{r_s} \]

\[ v^2 \leq v_s^2 \] \( r \leq r_s \)

\[ \frac{v}{v_s} \approx e^{-2\frac{r_s}{r}} = e^{-\frac{1}{H_r}} \]

I, 2: \( v^2 \leq v_s^2 \) \( r \geq r_s \)

\[ -\ln \frac{v^2}{v_s^2} \approx 4\ln \frac{r}{r_s} \]

\[ v \approx \frac{1}{r^2} \]

\( v^2 \geq v_s^2 \) \( r \leq r_s \)

\[ v \approx 2v_s r^{-1/2} \]
also possible: case II  

\[ v(r) \text{ exists for } 1 \leq r \leq \infty, \text{ but} \]

\[ r = r_s: \ v(r_s) = v_s \text{ and } \frac{dv^2}{dr} = 0 \]

2 possibilities

case II, 1: \[ v \geq v_s \text{ for } 1 \leq r \leq \infty, \ C > -3 \] supersonic solution

case II, 2: \[ v \leq v_s \text{ for } 1 \leq r \leq \infty, \ C < -3 \] subsonic solution

both cases not realized

however, II,2 was for a long time discussed as a possible option, until space observations showed that solar wind is supersonic
also not realized is **case III**
non-unique solutions

in \( 1 \leq r \leq r_s \)

with \( r = r_s : v(r_s) = v_s \) and \( \frac{dv^2}{dr} = \infty \)

or \( r_s \leq r \leq \infty \)

**note:** discussion of solution topologies crucial to identify physically realized solution. We will repeat this, when dealing with the much more complicated case of radiation driven winds.
Discussion of the realistic solution:
trans-sonic, monotonic, increasing (Parker, 1958)

\[
\frac{v^2}{v_s^2} - ln\frac{v^2}{v_s^2} = 4ln\frac{r}{r_s} + 4\frac{r_s}{r} - 3
\]

\[
\frac{v^2}{v_s^2} e^{-\frac{v^2}{v_s^2}} = \left(\frac{r_s}{r}\right)^4 e^{-4\frac{r_s}{r} + 3}
\]

\[
r = r_s: \quad \frac{dv^2}{v_s^2} = \frac{2}{r_s} = \frac{8v_s^2}{v_{esc}^2}
\]

\[
r_s = \frac{v_{esc}^2}{4v_s^2}
\]

limiting cases: 1< r << r_s, v << v_s:

\[
v = v_s \left(\frac{r_s}{r}\right)^2 e^{-\frac{2r_s}{r} + \frac{3}{2}}
\]

r >> r_s, v >> v_s:

\[
v = 2v_s \left\{ln\frac{r}{r_s}\right\}^{1/2}
\]

v \to \infty \text{ if } r \to \infty, \text{ consequence of the unrealistic assumption } T = \text{ const.}, \text{ i.e. pressure can accelerate until infinity in reality } T \text{ drops and } v \text{ remains finite}
density stratification

Eq. 4a \[ \dot{M} = 4\pi R_*^2 r^2 \rho v \]

\[ \rho(r) = \rho_1 v_1 \frac{1}{r^2 v} \]

Eq. 12

\[ \rho_1, v_1 \text{ at } r = 1 \text{ photosphere} \]

\[ \frac{\rho^2}{\rho_1^2} e^{-\frac{1}{r^4} \frac{\rho^2}{\rho_1^2} \frac{v}{v_s}^2} = \frac{1}{r_s^4} e^{4 \frac{r_s}{r} - 3} \]

for \( r \ll r_s \rightarrow \) hydrostatic solution, for \( r \ll r_s \rightarrow \rho < \rho_{\text{static}} \)

mass-loss rate

if \( \rho_1 \) at bottom of wind known \( \rightarrow \) with \( v_1 \) (eq. 11) \[ \dot{M} = 4\pi R_*^2 \rho_1 v_1 \]

for \( \rho_1 = 10^{-14} \text{ g/cm}^3 \)

<table>
<thead>
<tr>
<th>( T_{\text{corona}} )</th>
<th>( r_s )</th>
<th>( \dot{M} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( 10^6 \text{K} )</td>
<td>6.9</td>
<td>1.6 ( 10^{-14} ) ( M_\odot/yr )</td>
</tr>
<tr>
<td>2</td>
<td>3.5</td>
<td>3.7 ( 10^{-12} )</td>
</tr>
<tr>
<td>3</td>
<td>2.3</td>
<td>8.2 ( 10^{-11} )</td>
</tr>
</tbody>
</table>

observed value is \( 2 \ 10^{-13} \ M_\odot/yr \)
THE VELOCITY AND DENSITY DISTRIBUTION

\( \log \frac{v}{v_0} \) vs. \( \log \frac{r}{r_0} \)

\( \log \frac{\rho}{\rho_0} \)

velocity

density

.... hydrostatic solution

\( r_0 = \frac{r_3}{10} \)

from Lamers & Cassinelli
4. Additional forces: radiation

Pressure driven winds can explain solar wind (note: that in reality you need magnetic fields to explain coronal heating) but... can winds of hot stars be driven by pressure only?

Crucial observational facts:

- Cool winds, no corona: $T_{\text{wind}} \sim T_{\text{eff}} \sim 40000K$
  $v_s \sim 20 \text{ km/s}$
- Sonic point $v = v_s$ very close to photosphere at $r \sim 1.05$
- $v_{\text{esc}} \sim 500 \ldots 1000\text{km/s}$

With pressure driven wind theory

$$r_s^{\text{theory}} = \frac{v_{\text{esc}}^2}{4v_s^2} \approx 100...2500$$

We need additional force $g_{\text{add}}$ !!!
hot stars $B_v(T)$ large because $T$ large
$I_v, J_v$ large
large amount of photon momentum
photon absorption transfers momentum to atmospheric plasma

calculation of $g_{add} = g_{extra}$:

radiative acceleration

$grad = \frac{\text{absorbed momentum}}{\text{time} \cdot \text{mass}}$

absorbed energy per unit time:

momentum:

per time & mass:

only projection into radial direction contributes

$grad(\mu, \nu, r)d\omega d\nu = \frac{\kappa_v I_v}{c \rho} \mu d\nu d\omega$
with azimuthal symmetry of radiation field and \( d\omega = 2\pi d\mu \)

\[
g_{\text{rad}}(r) = \frac{2\pi}{c\rho(r)} \int_0^\infty \int_{-1}^1 \kappa_\nu I_\nu(r, \mu)d\mu d\nu
\]

Eq. 13

Eq. 13 exact, as long as all absorption processes included

\[
\kappa_\nu = n_E \sigma_E + \kappa_\nu^c + \kappa_\nu^L
\]

Eq. 14

\[
g_{\text{rad}}(r) = g_{\text{rad}}^{Th} + g_{\text{rad}}^c + g_{\text{rad}}^L
\]

Eq. 15

\[
g_{\text{rad}}^{Th} = \frac{4\pi}{c} \frac{n_e \sigma_E}{\rho} \int_0^\infty H_\nu d\nu = \frac{1}{c} \frac{n_E \sigma_E}{\rho} \frac{L}{4\pi r^2 R_*^2}
\]

\[
g_{\text{rad}}^c = \frac{4\pi}{c\rho} \int_0^\infty \kappa_\nu H_\nu d\nu
\]

\[
g_{\text{rad}}^L = \frac{4\pi}{c\rho} \int_0^\infty \int_{-1}^1 \kappa_\nu^L(\mu, v(r))\mu I(\mu)\nu d\mu d\nu
\]

remember:

\[
H_\nu = \frac{1}{2} \int_{-1}^1 \mu I_\nu(\mu) d\mu \quad H = \int_0^\infty H_\nu d\nu \quad L = 4\pi R_*^2 r^2 (4\pi H)
\]
corollary: approximation for $\tau_\nu \gg 1$ at all frequencies

radiation field thermalized at all frequencies

“diffusion approximation”

$$H_\nu = \frac{1}{2} \int_{-1}^{1} \mu I_\nu(\mu) d\mu = \frac{1}{3} \frac{1}{\kappa_\nu} \frac{dB_\nu(T)}{dr}$$

$$g_{rad}(r) = \frac{4\pi}{c\rho} \frac{d}{dr} \frac{1}{3} \int_0^\infty B_\nu(T) d\nu = \frac{4\pi}{c\rho} \frac{d}{dr} \frac{1}{3} \sigma_B \frac{d}{dr} T^4$$

$$g_{rad}(r) = \frac{1}{\rho} \frac{d}{dr} P_{rad} \quad P_{rad}(r) = \frac{1}{3} \frac{4\pi}{c} \sigma_B T^4(r)$$

“radiation pressure”
in hot star winds the influence of $g_{rad}^{c}$ is usually small but influence of Thomson scattering

$$g_{rad}^{Th} = \frac{1}{c} \frac{n_E \sigma_E}{\rho} \frac{L}{4\pi r^2 R_*^2}$$

with

$$n_E = n_p (1 + qY), \quad Y = \frac{n_{He}}{n_H}, \quad \rho = n_p m_p (1 + 4Y), \quad \frac{n_E}{\rho} = \frac{1}{m_p} \frac{1 + qY}{1 + 4Y}$$

q electrons provided by helium nucleus (0 ... 2)

$$g_{rad}^{Th} = \frac{\sigma_E}{c m_p} \frac{1 + qY}{1 + 4Y} \frac{L}{4\pi r^2 R_*^2}$$

and with

$$g(r) = G \frac{M_*}{R_*^2} \frac{1}{r^2}$$

$$\Gamma_{Th} = \frac{g_{rad}^{Th}}{g(r)} = 3.062 \cdot 10^{-5} \frac{1 + qY}{1 + 4Y} \frac{L}{L_\odot} \frac{M_\odot}{M_*}$$

Eq. 16

with Eq. 4b new momentum equation

$$\nu \frac{dv}{R_*dr} = -\frac{1}{\rho} \frac{dP_{gas}}{R_*dr} - \frac{GM_*}{R_*^2} \frac{1}{r^2} (1 - \Gamma_{Th}) + g_{rad}^L$$

Eq. 17

Thomson scattering reduces gravity by constant factor
with \( \tilde{v}_{esc}^2 = v_{esc}^2(1 - \Gamma_{Th}) \)  

Eq. 18 effective escape velocity

and eq. 6 we obtain new equation of motion

\[
\left\{1 - \frac{v_s^2}{v^2}\right\}v \frac{dv}{dr} = \frac{1}{r} \left\{2v_s^2 - \frac{1}{2} \tilde{v}_{esc}^2 \frac{1}{r}\right\} - \frac{dv_s^2}{dr} + R_L g_{rad}^L
\]

Eq. 19

does Thomson scattering alone do the job? if we ignore \( g_{rad}^L \), \( \frac{dv_s}{dr} \) sonic point at

\[
r_s = \frac{\tilde{v}_{esc}}{4v_s^2} = \frac{v_{esc}^4}{4v_s^4}(1 - \Gamma_{Th})
\]

Eq. 19b

typical \( \Gamma \)-value for hot supergiants \( \Gamma = 0.5 \)

sonic point for hot stars still at \( r_s \sim 50\ldots100 \), but observed is \( r_s \sim 1.1 \)

need additional accelerating force!!!!
5. Very simplified description of line driven winds

As first step: very simplified description of assumptions:

- only radially streaming photons
- only lines with $\tau_s \gg 1$ contribute
- Sobolev approximation $\Delta \nu \gg v_{th}$, $\Delta \nu \approx \frac{\Delta v}{c} \nu_i \gg \Delta \nu_D$
- number of strong lines independent of $r$
shell in stellar wind: calculation of line $i$ at $\nu_i \Rightarrow g^{i}_{\text{rad}}$

\[
g^{i}_{\text{rad}} = \frac{\text{abs. momentum}}{\Delta t \cdot \Delta m} = \frac{\frac{L L_{\nu_i} \Delta \nu_i}{c L}}{\Delta \nu_i \gg \Delta \nu_D}
\]

\[
\Delta m = 4\pi r^2 \rho \Delta r R_*^3
\]

\[
\Delta \nu_i = \frac{\Delta \nu}{c} \nu_i, \quad \Delta \nu_i \gg \Delta \nu_D
\]

\[
g^{i}_{\text{rad}} = \frac{L L_{\nu_i} \Delta \nu_i}{c^2 L} \frac{1}{4\pi r^2 \rho R_*^3} \frac{1}{dr} {dv}
\]

\[
g^{L}_{\text{rad}} = \sum_i g^{i}_{\text{rad}}
\]

contribution by all lines:

\[
g^{L}_{\text{rad}} = \frac{L}{c^2} \sum_i \frac{L_{\nu_i} \Delta \nu_i}{L} \frac{1}{4\pi r^2 \rho R_*^3} \frac{1}{dr} {dv}
\]

Eq. 20

line acceleration ~ velocity gradient !!!

effective number of strong lines $N_{eff}$
new equation of motion \((T = \text{const.})\)

\[
\left\{ 1 - \frac{v_s^2}{v^2} \right\} v \frac{dv}{dr} = \frac{1}{r} \left\{ 2v_s^2 - \frac{1}{2} \frac{\tilde{v}_{esc}^2}{r} \right\} + \frac{L}{c^2 N_{eff}} \frac{1}{4\pi R_*^2 r^2 \rho} \frac{dv}{dr} \quad \text{Eq. 21a}
\]

with

\[
\dot{M} = 4\pi R_*^2 r^2 \rho v
\]

\[
\left\{ 1 - \frac{v_s^2}{v^2} - \frac{L}{c^2 M} \right\} \frac{dv^2}{v_s^2} = \frac{4}{r} \left\{ 1 - \frac{1}{4} \frac{\tilde{v}_{esc}^2}{v_s^2} \frac{1}{r} \right\} \quad \text{Eq. 21b}
\]

because \(g_{rad}^L \propto \frac{dv^2}{dr}\) no singularity at \(v = v_s\)

r.h.s. of Eq. 21b negative smaller than old \(r_s\)

we assume \(r_s < \frac{1}{4} \frac{\tilde{v}_{esc}}{v_s^2}\)

and prove later that this is correct
multiplying eq. 21a by \(4\pi R_*^2 r^2 \rho\) and assuming \(v^2 \leq v_s^2\)

\[
\dot{M} \frac{dv}{dr} = \frac{L}{c^2} N_{eff} \frac{dv}{dr} - \frac{1}{2} \tilde{v}_{esc}^2 4\pi R_*^2 \rho
\]

\[
\dot{M} = \frac{L}{c^2} N_{eff} \left\{1 - \frac{1}{2} \tilde{v}_{esc}^2 4\pi R_*^2 \rho c^2 \frac{dv}{dr} L N_{eff} \right\}
\]

\[
\epsilon \geq 0 \implies \epsilon = \text{const.} \frac{dv}{dr} \geq 0 \implies \epsilon \geq 0
\]

\[
\dot{M} = \frac{L}{c^2} N_{eff} \{1 - \epsilon\}
\]

Eq. 21c

0 \leq \epsilon \leq 1

combining eq. 21b and c for \(r \gg r_s\)

\[
\left\{1 - \frac{1}{1 - \epsilon}\right\} \frac{dv^2}{dr} = -\tilde{v}_{esc}^2 \frac{1}{r^2}
\]
\[ \frac{dv^2}{dr} = \frac{1 - \epsilon}{\epsilon} \tilde{v}_{esc}^2 \frac{1}{r^2} \]

integrating from \( r_s \) to \( r \)

\[ v^2(r) = \frac{1 - \epsilon}{\epsilon} \tilde{v}_{esc}^2 \left( \frac{1}{r_s} - \frac{1}{r} \right) + v_s^2 \]

\[ v_\infty^2 = \frac{1 - \epsilon}{\epsilon} \tilde{v}_{esc}^2 \]

from observations (Abbott, 1978, 1982)

\[ v_\infty = 3v_{esc} \implies \epsilon \approx 0.1 \]

\[ \dot{M} = \frac{L}{c^2} N_{eff}, \quad N_{eff}^{\text{theory}} \approx 50 \]

\[ \dot{M}[M_\odot/yr] = 3.5 \cdot 10^{-12} \frac{L}{L_\odot} \]

Eq. 22 \( \beta \)-velocity field, \( \beta = 0.5 \)

Eq. 23
$V_\infty = 3 \tilde{V}_{\text{esc}}$

- $\tilde{V}_{\text{esc}} = \sqrt{\frac{2GM}{R_x} \left( \frac{1 - \Gamma}{\epsilon} \right)^{1/2}}$

$V_\infty$ = luminosity class I or f
$\bullet$ = luminosity class V - II
\[ \log T = -11.84 + 1.62 \log L/L_\odot \]

\[ \log f_i = -11.45 + \log L/L_\odot \]
Simplified theory of line driven winds explains

- \( v_\infty \propto v_{esc} \) but does not give proportionality constant
- right order of magnitude for \( \dot{M} \)
  - but predicts \( \dot{M} \propto L \)
  - whereas is observed \( \dot{M} \propto L^x, \ x = 1.7 \)

improved theory needed!!!