2. Basic assumptions for stellar atmospheres

1. geometry, stationarity
2. conservation of momentum, mass
3. conservation of energy
4. Local Thermodynamic Equilibrium
1. Geometry

Stars as gaseous spheres $\rightarrow$ **spherical symmetry**

Exceptions: rapidly rotating stars

Be stars $v_{\text{rot}} = 300 - 400$ km/s
(Sun $v_{\text{rot}} = 2$ km/s)

For stellar photospheres typically: $\Delta r/R \ll 1$

Sun: $R_\odot = 700,000$ km

- Photosphere $\Delta r = 300$ km $\quad \Delta r/R = 4 \times 10^{-4}$
- Chromosphere $\Delta r = 3000$ km $\quad \Delta r/R = 4 \times 10^{-3}$
- Corona $\quad \Delta r/R \approx 3$
As long as $\Delta r/R \ll 1$: plane-parallel symmetry

consider a light ray through the atmosphere:

$\Delta r/R \ll 1$

$\Delta r/R \sim 1$

Plane-parallel symmetry
very small curvature ($\alpha \sim \beta$)
Solar photo/cromosphere, dwarfs, (giants)

Spherical symmetry
significant curvature ($\alpha \neq \beta$)
Solar corona, supergiants, expanding envelopes (OBA stars), novae, SN
Homogeneity & stationarity

We assume the atmosphere to be **homogeneous**

**Counter-examples:**
- sunspots, granulation, non-radial pulsations
- clumps and shocks in hot star winds
- magnetic Ap stars

and **stationary**

Most spectra are time-independent: $\partial/\partial t = 0$ *(if we don’t look carefully enough…)*

Exceptions: explosive phenomena (SN), stellar pulsations, magnetic stars, mass transfer in close binaries
2. Conservation of momentum & mass

Consider a mass element $dm$ in a spherically symmetric atmosphere.

The acceleration of the mass element results from the sum of all forces acting on $dm$, according to Newton’s 2nd law:

$$dm \frac{d}{dt} v(r,t) = \sum_i df_i = F$$
a. Hydrostatic equilibrium

Assuming a hydrostatic stratification: \( v(r,t) = 0 \)

\[
dm \frac{d}{dt} v(r,t) = \sum_i df_i = F
\]

\[
0 = \sum_i df_i
\]

Gravitational forces:

\[
df_{grav} = -G \frac{M_r \cdot dm}{r^2} = -g(r) \, dm
\]

Gas Pressure forces:

\[
df_{p,gas} = -A P(r + dr) + A P(r) = -A \frac{dP}{dr} \, dr
\]

Radiation forces:

\[
df_{rad} = g_{rad}(r) \, dm = \frac{\text{absorbed photon momentum}}{dt}
\]
In equilibrium:

\[ 0 = \sum_i df_i = -g(r)dm - A \frac{dP}{dr} dr + g_{rad} dm \]

and substituting \( dm = A \rho dr \):

\[ -g(r)\rho A dr - A \frac{dP}{dr} dr + g_{rad} \rho A dr = 0 \]

\[ \frac{dP}{dr} = -\rho(r)[g(r) - g_{rad}] \]

Hydrostatic equilibrium in spherical symmetry
Approximation for \( g(r) \)

The mass within the atmosphere \( M(r) \) – \( M(R) \) \( \ll \) \( M(R) = M_* \)

\[
M(R) = M(r) = M_*
\]

\[
g(r) = G \frac{M_*}{r^2}
\]

Example: take a geometrically thin photosphere

\[
\Delta M_{\text{phot}} \approx \frac{-4\pi}{3} \left[ (R + \Delta r)^3 - R^3 \right] = \frac{-4\pi}{3} \left[ R^3 \left(1 + \frac{\Delta r}{R}\right)^3 - R^3 \right]
\]

\[
\approx \frac{-4\pi}{3} R^3 \left[1 + 3 \frac{\Delta r}{R} - 1\right] = \frac{-4\pi}{3} 4\pi R^2 \Delta r
\]
Example: the sun

\[ R = 7 \times 10^{10} \text{ cm}, \Delta r = 3 \times 10^7 \text{ cm}, \rho \sim m_H N = 1.7 \times 10^{-24} \times 10^{15} \text{ g/cm}^3: \]

\[ \Delta M_{\text{phot}} = 3 \times 10^{21} \text{ g} \]

\[ \Delta M_{\text{phot}}/M = 10^{-12} \]

Moreover in plane-parallel symmetry:

\[ \Delta r/R \ll 1 \]

\[ g(r) = \text{const} \]

\[ g = G M/\text{R}^2 \]

<table>
<thead>
<tr>
<th>Star Type</th>
<th>log ( g ) (cgs: cm/s²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Main sequence</td>
<td>4</td>
</tr>
<tr>
<td>supergiant</td>
<td>3.5 – 0.8</td>
</tr>
<tr>
<td>white dwarf</td>
<td>8</td>
</tr>
<tr>
<td>Sun</td>
<td>4.44</td>
</tr>
<tr>
<td>Earth</td>
<td>3.0</td>
</tr>
</tbody>
</table>
Assume:
• plane-parallel symmetry
• \( g_{\text{rad}} \ll g \)
• ideal gas:

\[
P = \frac{\rho k T}{m_H \mu} = \frac{\rho R T}{\mu} = N_g k T
\]

Note:
\[
\frac{kT}{m_H \mu} = v_{\text{sound}}^2
\]

and:
\[
\frac{1}{\rho} \frac{d\rho}{dr} \gg \frac{1}{T} \frac{dT}{dr}
\]

\( k \): Boltzmann constant = 1.38 x 10^{-16} \text{ erg K}^{-1}

\( m_H \): mass of H atom = 1.66 x 10^{-24} \text{ g}

\( \mu \): mean molecular weight per free particle = \(<m>/m_H \)

\( R = k N_A = 8.314 \times 10^7 \text{ erg/mole/K} \)

\[
\frac{dP}{dr} = -\rho(r)g(r)
\]
Barometric formula

\[ P = \frac{\rho kT}{m_H \mu} \]

\[ \frac{dP}{dr} = -\rho(r)g(r) \]

\[ \frac{k}{m_H \mu} \left[ T \frac{d\rho}{dr} + \rho \frac{dT}{dr} \right] = -g \rho \]

Negligible by assumption

\[ \frac{1}{\rho} \frac{d\rho}{dr} = -\frac{g m_H \mu}{kT} \]

and:

\[ H = \frac{kT}{g m_H \mu} \]

Pressure scale height

\[ H(Sun) = 150 \text{ km} \]

Solution:

\[ \rho(r) = \rho(r_0) e^{-\frac{r-r_0}{H}} \]
Approximate solution: Barometric formula

\[ \frac{dP}{dr} = -\rho(r)g(r) \]

\[ \rho(r) = \rho(r_0) e^{\frac{r-r_0}{H}} \]

- exponential decline of density
- scale length \( H \sim \frac{T}{g} \)
- explains why atmospheres are so thin

\[ H = \frac{kT}{g m_H \mu} \]
Let us now consider the case in which there are deviations from hydrostatic equilibrium and $v(r,t) \neq 0$.

Newton’s law:

$$ dm \frac{d}{dt} v(r,t) = \sum_i df_i $$

$$ \frac{d}{dt} v(r,t) = \lim_{\Delta t \to 0} \frac{v(r + \Delta r, t + \Delta t) - v(r,t)}{\Delta t} $$

Taylor expansion:

$$ v(r + \Delta r, t + \Delta t) = v(r,t) + \frac{\partial v}{\partial r} \Delta r + \frac{\partial v}{\partial t} \Delta t $$

and

$$ \Delta r = v \Delta t $$

In general:

$$ \frac{d}{dt} \alpha(r,t) = \frac{\partial}{\partial t} \alpha(r,t) + v(r,t) \times \text{grad}\{\alpha(r,t)\} $$
For the assumption of stationarity $\frac{\partial}{\partial t} = 0$

$$\frac{d}{dt} v(r, t) = v \frac{\partial v}{\partial r}$$

$$dm \frac{d}{dt} v(r, t) = dm v \frac{\partial v}{\partial r} = \sum_i df_i$$

$$dm = A \rho dr$$

$$- g(r) dm - A \frac{dP}{dr} dr + g_{rad} dm$$

$$\frac{dP}{dr} = - \rho [g(r) - g_{rad} + v \frac{dv}{dr}]$$

new term
The equation of continuity, mass conservation

\[ \dot{M} = \frac{d}{dt} M = 4\pi r^2 \rho v \]

\[ v = \frac{\dot{M}}{4\pi r^2 \rho} \]

From:

\[ \frac{dP}{dr} = -\rho[g(r) - g_{rad}] + v \frac{dv}{dr} \]

and from the equation of state:

\[ \frac{dP}{dr} = \frac{kT}{m_H \mu} \]

\[ \frac{d\rho}{dr} = \frac{\rho}{v_{sound}} \frac{d\rho}{dr} \]

\[ \rho v \frac{dv}{dr} = -2\rho \frac{v^2}{r} - v^2 \frac{d\rho}{dr} \]

Assuming again that the temperature gradient is small
When $v << v_{\text{sound}}$: practically hydrostatic solution ($v = 0$) for density stratification. This is reached well below the sonic point (where $v = v_{\text{sound}}$).

example: $v_{\text{sound}} = 6$ km/s for solar photosphere, 20 km/s for O stars
$v_{\text{esc}} = 100$ to 1000 km/s for main sequence and supergiant stars

$\Rightarrow (v_{\text{sound}}/v_{\text{esc}})^2 << 1$
3. Conservation of energy

Stellar interior: production of energy via nuclear reactions
Stellar atmosphere: negligible production of energy

the energy flux is conserved at any given radius

\[ F(r) = \frac{\text{energy}}{\text{area} \cdot \text{time}} \]

\[ 4\pi r^2 F(r) = \text{const} = \text{luminosity} \quad L \]

In spherical symmetry: \( r^2 F(r) = \text{const} \)

\[ F(r) \sim \frac{1}{r^2} \]

Plane-parallel: \( r^2 \sim R^2 \sim \text{const} \)

\[ F(r) \sim \text{const} \]
4. Concepts of Thermodynamics

Radiation field

Consider a closed “cavity” in thermodynamic equilibrium TE (photons and particles in equilibrium at some temperature T).

The specific intensity emitted (through a small hole) is (energy per area, per unit time, per unit frequency, per unit solid angle) (#Planck function#):

\[
I_v \, dv = B_v(T) \, dv = \frac{2h \nu^3}{c^2} \frac{1}{e^{hv/kT} - 1} \, dv \quad \text{erg cm}^{-2} \text{s}^{-1} \text{Hz}^{-1} \text{sr}^{-1}
\]

**black body radiation:** universal function dependent only on T not on chemical composition, direction, place, etc.

**Note:** stars don’t radiate as blackbodies, since they are not closed systems !!! But their radiation follows the Planck function qualitatively.
properties of Planck function

a. $B_\nu(T_1) > B_\nu(T_2)$ (monotonic) for every $T_1 > T_2$ (no function crossing)

b. $\nu_{\text{max}} / T = \text{const}$

Wien’s displacement law

$$\lambda_{\text{max}} = \frac{2.898 \times 10^{-7}}{T} \text{Å for } B_\lambda$$

$$\lambda_{\text{max}} = \frac{5.099 \times 10^{-7}}{T} \text{Å for } B_\nu$$

c. Stefan-Boltzmann law: (total flux)

$$F = \pi \int_0^\infty B_\nu(T) d\nu = \sigma T^4$$

$$\sigma = 5.67 \times 10^{-5} \text{ erg cm}^{-2} \text{s}^{-1} \text{K}^{-4}$$

$h\nu/kT >> 1$: Wien

$$B_\nu(T) \approx \frac{2h\nu^3}{c^2} e^{-h\nu/kT}$$

$h\nu/kT << 1$: Rayleigh-Jeans

$$B_\nu(T) \approx \frac{2h\nu^2}{c^2} kT$$

Rybicki & Lightman, 79
Concepts of Thermodynamics

Gas particles

1. Velocity distribution

In complete equilibrium this is given by Maxwell distribution (needed, e.g., for collisional rates):

\[
f(v)dv = 4\pi v^2 \left( \frac{m}{2\pi kT} \right)^{3/2} e^{-\frac{mv^2}{2kT}} dv
\]

most probable speed: \( v_p = (2kT/m)^{1/2} \)

average speed: \( \langle v \rangle = (8kT/\pi m)^{1/2} \)

r.m.s. speed: \( v_{rms} = (3kT/m)^{1/2} \)
2. Energy level distribution

Boltzmann’s equation for the population density of excited states in TE

\[
\frac{n_u}{n_l} = \frac{g_u}{g_l} e^{\frac{(E_u-E_l)}{kT}}
\]

\(g_u, g_l\): statistical weights = 2J+1: number of degenerate states

= 2n^2 for Hydrogen

\[
\frac{n_n}{n_{tot}} = \frac{g_n}{u} e^{\frac{E_n}{kT}}
\]

\[u = \sum_i g_i e^{\frac{E_i}{kT}}\]  
partition function
3. Kirchhoff’s law

For a thermal emitter in TE

\[
\frac{\varepsilon_v}{\kappa_v} = \frac{\text{emission coefficient}}{\text{absorption coefficient}} = B_v(T)
\]

as a consequence of Boltzmann’s equation for excitation and Maxwell’s velocity law.
Local Thermodynamic Equilibrium

A star as a whole (or a stellar atmosphere) is far from being in thermodynamic equilibrium: energy is transported from the center to the surface, driven by a temperature gradient.

But for sufficiently small volume elements $dV$, we can assume TE to hold at a certain temperature $T(r)$.

**Local Thermodynamic Equilibrium (LTE)**

- **good approx**: stellar interior (high density, small distance travelled by photons, nearly-isotropic, thermalized radiation field)
- **bad approx**: gaseous nebulae (low density, non-local radiation field, optically thin)
- stellar atmospheres: optically thick but moderate density
Local Thermodynamic Equilibrium

1. Energy distribution of the gas
determined by local temperature $T(r)$
   appearing in Maxwell’s/ Boltzmann’s equations, and Kirchhoff’s law

2. Energy distribution of the photons
photons are carriers of non-local energy through the atmosphere
   $\Rightarrow I_\nu(r) \neq B_\nu[T(r)]$
   $I_\nu$ is a superposition of Planck functions originating at different
   depths in the atmosphere $\Rightarrow$ radiation transfer
LTE vs NLTE

**LTE**

- Each volume element separately in thermodynamic equilibrium at temperature $T(r)$

1. $f(v) \ dv = \text{Maxwellian with } T = T(r)$
2. Saha: $(n_p \ n_e)/n_1 \sim T^{3/2} \exp(-h\nu_1/kT)$
3. Boltzmann: $n_i/n_1 = g_i/g_1 \exp(-h\nu_{1i}/kT)$

However:

- Volume elements not closed systems, interactions by photons
- LTE non-valid if absorption of photons disrupts equilibrium
LTE vs NLTE

NLTE if

rate of photon absorptions $\gg$ rate of electron collisions

$$I_v (T) \sim T^\alpha, \alpha > 1 \quad \Rightarrow \quad n_e T^{1/2}$$

**LTE**

*valid*: low temperatures & high densities

*non-valid*: high temperatures & low densities
LTE vs NLTE in hot stars

Kudritzki 1978