3. Transport of energy: radiation

- specific intensity, radiative flux
- optical depth
- absorption & emission
- equation of transfer, source function
- formal solution, limb darkening
- temperature distribution
- grey atmosphere, mean opacities
Energy flux conservation

No sinks and sources of energy in the atmosphere
→ all energy produced in stellar interior is transported through the atmosphere
→ at any given radius $r$ in the atmosphere:

$$4\pi r^2 F(r) = \text{const.} = L$$

$F$ is the energy flux per unit surface and per unit time. Dimensions: [erg/cm$^2$/sec]
The energy transport is sustained by the temperature gradient.
The steepness of this gradient is dependent on the effectiveness of the energy transport through the different atmospheric layers.
Mechanisms of energy transport

a. **radiation**: $F_{\text{rad}}$ (most important)

b. **convection**: $F_{\text{conv}}$ (important especially in cool stars)

c. **heat production**: e.g. in the transition between solar cromosphere and corona

d. **radial flow of matter**: corona and stellar wind

e. **sound waves**: cromosphere and corona

We will be mostly concerned with the first 2 mechanisms: $F(r)=F_{\text{rad}}(r) + F_{\text{conv}}(r)$. In the outer layers, we always have $F_{\text{rad}} \gg F_{\text{conv}}$
The specific intensity

Measures of energy flow: **Specific Intensity** and **Flux**

The amount of energy $dE_\nu$ transported through a surface area $dA$ is proportional to $dt$ (length of time), $dv$ (frequency width), $d\omega$ (solid angle) and the projected unit surface area $\cos \theta dA$.

The proportionality factor is the **specific Intensity** $I_\nu(\cos \theta)$

$$dE_\nu = I_\nu(\cos \theta) \cos \theta dA d\omega dv dt$$

([I$\nu$]: erg cm$^{-2}$ sr$^{-1}$ Hz$^{-1}$ s$^{-1}$)

$$I_\lambda = \frac{c}{\lambda^2} I_\nu$$

(from $I_\lambda d\lambda = I_\nu dv$ and $\nu = c/\lambda$)

Intensity depends on location in space, direction and frequency.
Invariance of the specific intensity

The area element $dA$ emits radiation towards $dA'$. In the absence of any matter between emitter and receiver (no absorption and emission on the light paths between the surface elements) the amount of energy emitted and received through each surface elements is:

$$dE_\nu = I_\nu(\cos \theta) \cos \theta dA d\omega d\nu dt$$
$$dE'_\nu = I'_\nu(\cos \theta') \cos \theta' dA' d\omega' d\nu dt$$
Invariance of the specific intensity

Specific intensity is constant along rays - as long as there is no absorption and emission of matter between emitter and receiver.

\[
\frac{d\omega}{\text{distance}^2} = \frac{dA' \cos \theta'}{r^2}
\]

\[
d\omega' = \frac{dA \cos \theta}{r^2}
\]

\[
\begin{align*}
    dE_\nu &= I_\nu (\cos \theta) \cos \theta \, dA \, d\omega \, d\nu \, dt \\
    dE'_\nu &= I'_\nu (\cos \theta') \cos \theta' \, dA' \, d\omega' \, d\nu \, dt
\end{align*}
\]

\[
I_\nu = I'_\nu
\]

\[
\text{In TE: } I_\nu = B_\nu
\]
Spherical coordinate system and solid angle $d\omega$

Solid angle: 

$$d\omega = \frac{dA}{r^2}$$

Total solid angle: 

$$\frac{4\pi r^2}{r^2} = 4\pi$$

$$dA = (r \, d\theta)(r \, \sin \theta \, d\phi)$$

$$\rightarrow d\omega = \sin \theta \, d\theta \, d\phi$$

Define $\mu = \cos \theta$

$$d\mu = - \sin \theta \, d\theta$$

$$d\omega = \sin \theta \, d\theta \, d\phi = -d\mu \, d\phi$$
Radiative flux

How much energy flows through surface element $dA$?

$$dE_\nu \sim I_\nu \cos \theta \ d\omega$$

→ integrate over the whole solid angle ($\Omega = 4\pi$):

$$\pi F_\nu = \int I_\nu (\cos \theta) \cos \theta \ d\omega = \int_0^{2\pi} \int_0^{\pi} I_\nu (\cos \theta) \cos \theta \sin \theta \ d\theta \ d\phi$$

"astrophysical flux" $\frac{4\pi}{\pi}$

$F_\nu$ is the **monochromatic radiative flux**.

The factor $\pi$ in the definition is historical.

$F_\nu$ can also be interpreted as the net rate of energy flow through a surface element.
Radiative flux

The monochromatic **radiative flux** at frequency \( \nu \) gives the net rate of energy flow through a surface element.

\[
dE_{\nu} \sim I_{\nu} \cos \theta \, d\omega \rightarrow \text{integrate over the whole solid angle } (\Omega = 4\pi):
\]

\[
\pi F_{\nu} = \int_{4\pi} I_{\nu}(\cos \theta) \cos \theta \, d\omega = \int_{0}^{2\pi} \int_{0}^{\pi} I_{\nu}(\cos \theta) \cos \theta \sin \theta d\theta d\phi
\]

We distinguish between the outward direction \((0 < \theta < \pi/2)\) and the inward direction \((\pi/2 < \theta < \pi)\), so that the net flux is:

\[
\pi F_{\nu} = \pi F_{\nu}^+ - \pi F_{\nu}^- = \]

\[
= \int_{0}^{2\pi} \int_{0}^{\pi/2} I_{\nu}(\cos \theta) \cos \theta \sin \theta d\theta d\phi + \int_{0}^{2\pi} \int_{\pi/2}^{\pi} I_{\nu}(\cos \theta) \cos \theta \sin \theta d\theta d\phi
\]

**Note:** for \( \pi/2 < \theta < \pi \rightarrow \cos \theta < 0 \rightarrow \text{second term negative}!!\)
Total radiative flux

Integral over frequencies $\nu \rightarrow$

$$\int_{0}^{\infty} \pi F_{\nu} d\nu = F_{rad}$$

$F_{rad}$ is the total radiative flux.

It is the total net amount of energy going through the surface element per unit time and unit surface.
Stellar luminosity

At the outer boundary of atmosphere \( r = R_o \) there is no incident radiation

\[ \Rightarrow \text{Integral interval over } \theta \text{ reduces from } [0, \pi] \text{ to } [0, \pi/2]. \]

\[
\pi F_\nu(R_o) = \pi F^+_\nu(R_o) = \int_0^{2\pi} \int_0^{\pi/2} I_\nu(\cos \theta) \cos \theta \sin \theta d\theta d\phi
\]

This is the monochromatic energy that each surface element of the star radiates in all directions

If we multiply by the total stellar surface \( 4\pi R_o^2 \)

\[ \Rightarrow \text{monochromatic stellar luminosity at frequency } \nu \]

and integrating over \( \nu \)

\[ \Rightarrow \text{total stellar luminosity} \]

\[
4\pi R_o^2 \cdot \int_0^\infty \pi F^+_\nu(R_o) d\nu = L \quad \text{(Luminosity)}
\]
Observed flux

What radiative flux is measured by an observer at distance $d$?

→ integrate specific intensity $I_\nu$ towards observer over all surface elements

note that only half sphere contributes

\[
E_\nu = \int_{1/2 \text{ sphere}} dE = \Delta \omega \Delta \nu \Delta t \int_{1/2 \text{ sphere}} I_\nu(\cos \theta) \cos \theta \, dA
\]

in spherical symmetry: $dA = R_o^2 \sin \theta \, d\theta \, d\phi$

→ $E_\nu = \Delta \omega \Delta \nu \Delta t R_o^2 \int_0^{2\pi} \int_0^{\pi/2} I_\nu(\cos \theta) \cos \theta \sin \theta \, d\theta \, d\phi$

→ because of **spherical symmetry** the integral of intensity towards the observer over the stellar surface is proportional to $\pi F_\nu^+$, the flux emitted into all directions by one surface element !!
Observed flux

Solid angle of telescope at distance $d$:

$$\Delta \omega = \Delta A/d^2$$

$$E_\nu = \Delta \omega \Delta \nu \Delta t \frac{R_o^2}{d^2} \pi F_\nu^+(R_o)$$

$$F_{\nu}^\text{obs} = \frac{\text{radiative energy}}{\text{area} \cdot \text{frequency} \cdot \text{time}} = \frac{R_o^2}{d^2} \pi F_\nu^+(R_o)$$

This, and not $I_\nu$, is the quantity generally measured for stars. For the Sun, whose disk is resolved, we can also measure $I_\nu$ (the variation of $I_\nu$ over the solar disk is called the limb darkening).

$$\int_0^\infty F_{\nu}^\text{obs} \, d\nu = 1.36 \text{ KW/m}^2$$

Unlike $I_\nu$, $F_\nu$ decreases with increasing distance.
Mean intensity, energy density & radiation pressure

Integrating over the solid angle and dividing by $4\pi$:

$$J_\nu = \frac{1}{4\pi} \int I_\nu \, d\omega$$

mean intensity

$$u_\nu = \frac{\text{radiation energy}}{\text{volume}} = \frac{1}{c} \int I_\nu \, d\omega = \frac{4\pi}{c} J_\nu$$

energy density

$$p_\nu = \frac{1}{c} \int I_\nu \cos^2 \theta \, d\omega$$

radiation pressure
(important in hot stars)

pressure = \frac{\text{force}}{\text{area}} = \frac{d \text{momentum}(= E/c)}{dt} \cdot \frac{1}{\text{area}}
Moments of the specific intensity

\[ J_\nu = \frac{1}{4\pi} \int I_\nu \, d\omega = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_{-1}^{1} I_\nu \, d\mu = \frac{1}{2} \int_{-1}^{1} I_\nu \, d\mu \]

0th moment

for azimuthal symmetry

\[ H_\nu = \frac{1}{4\pi} \int I_\nu \cos \theta \, d\omega = \frac{1}{2} \int_{-1}^{1} I_\nu \, d\mu = \frac{F_\nu}{4} \]

1st moment

(Eddington flux)

\[ K_\nu = \frac{1}{4\pi} \int I_\nu \cos^2 \theta \, d\omega = \frac{1}{2} \int_{-1}^{1} I_\nu \, d\mu = \frac{c}{4\pi} p_\nu \]

2nd moment
Interactions between photons and matter

absorption of radiation

Over a distance $s$:

$I_\nu^o$ $\rightarrow$ $s$ $\rightarrow$ $I_\nu(s)$

$dI_\nu = -\kappa_\nu I_\nu ds$

$\kappa_\nu$: absorption coefficient

$[\kappa_\nu] = \text{cm}^{-1}$

loss of intensity in the beam (true absorption/scattering)

Con convention: $\tau_\nu = 0$ at the outer edge of the atmosphere, increasing inwards

Optical depth

$\tau_\nu := \int_0^s \kappa_\nu ds$

or: $d\tau_\nu = \kappa_\nu ds$
optical depth

\[ I_\nu(s) = I_\nu^0 e^{-\tau_\nu} \]

if \( \tau_\nu = 1 \rightarrow I_\nu = \frac{I_\nu^0}{e} \approx 0.37 I_\nu^0 \)

The optical thickness of a layer determines the fraction of the intensity passing through the layer.

The quantity \( \tau_\nu = 1 \) has a geometrical interpretation in terms of mean free path of photons \( \bar{s} \):

\[ \tau_\nu = 1 = \int_0^{\bar{s}} \kappa_\nu \, ds \]

We can see through atmosphere until \( \tau_\nu \approx 1 \)

optically thick (thin) medium: \( \tau_\nu > (\leq) 1 \)

photons travel on average for a length \( \bar{s} \) before absorption.
What is the average distance over which photons travel?

$$< \tau_\nu > = \int_0^\infty \tau_\nu p(\tau_\nu) d\tau_\nu$$

- probability that photon is absorbed: $$p(0, \tau_\nu) = \frac{\Delta I(\tau)}{I_0} = \frac{I_0 - I(\tau_\nu)}{I_0} = 1 - \frac{I(\tau_\nu)}{I_0}$$

- probability that photon is not absorbed: $$1 - p(0, \tau_\nu) = \frac{I(\tau_\nu)}{I_0} = e^{-\tau_\nu}$$

- probability that photon is absorbed in $$[\tau_\nu, \tau_\nu + d\tau_\nu]$$: $$p(\tau_\nu, \tau_\nu + d\tau_\nu) = \frac{dI_\nu}{I(\tau_\nu)} = d\tau_\nu$$

total probability: $$e^{-\tau_\nu} d\tau_\nu$$
photon mean free path

\[
< \tau_\nu > = \int_0^\infty \tau_\nu p(\tau_\nu) d\tau_\nu = \int_0^\infty \tau_\nu e^{-\tau_\nu} d\tau_\nu = 1
\]

\[
\int xe^{-x} \, dx = -(1 + x) e^{-x}
\]

mean free path corresponds to \(<\tau_\nu>=1\)

if \(\kappa_\nu(s) = \text{const}\) : 
\[
\Delta \tau_\nu = \kappa_\nu \Delta s \rightarrow \Delta s = \tilde{s} = \frac{1}{\kappa_\nu}
\]

(homogeneous material)
Principle of line formation

\[ T(\text{cont}) > T(\text{line}) \]

observer sees through the atmospheric layers up to \( \tau_v \approx 1 \)

In the continuum \( \kappa_v \) is smaller than in the line \( \rightarrow \) see deeper into the atmosphere
radiative acceleration

In the absorption process photons release momentum $E/c$ to the atoms, and the corresponding force is:

$$\text{force} = df_{\text{phot}} = \frac{\text{momentum} (= E/c)}{dt}$$

The infinitesimal energy absorbed is:

$$dE_{\nu}^{\text{abs}} = dI_{\nu} \cos \theta dA d\omega dt d\nu = \kappa_{\nu} I_{\nu} \cos \theta dA d\omega dt d\nu d\omega$$

The total energy absorbed is (assuming that $\kappa_{\nu}$ does not depend on $\omega$):

$$E^{\text{abs}} = \int_{0}^{\infty} \kappa_{\nu} \int_{4\pi} I_{\nu} \cos \theta d\omega d\nu \ dA dt ds = \pi \int_{0}^{\infty} \kappa_{\nu} F_{\nu} d\nu \ dA dt ds$$
Radiative acceleration

\[
\frac{df_{\text{phot}}}{c} = \frac{\pi}{\kappa_{\nu} f_{\nu}} \int_{0}^{\nu} \kappa_{\nu} F_{\nu} d\nu
dA dt ds = g_{\text{rad}} dm \quad (dm = \rho dA ds)
\]

\[
g_{\text{rad}} = \frac{\pi}{c \rho} \int_{0}^{\infty} \kappa_{\nu} F_{\nu} d\nu
\]
emission of radiation

dV = dA \; ds

dV = dA \; ds

energy added by emission processes within dV

\[ dE_{\nu}^{em} = \epsilon_{\nu} \; dV \; d\omega \; d\nu \; dt \]

\( \epsilon_{\nu} \) : emission coefficient

[\epsilon_{\nu}] = \text{erg cm}^{-3} \text{ sr}^{-1} \text{ Hz}^{-1} \text{ s}^{-1}
The equation of radiative transfer

If we combine absorption and emission together:

\[ dE^\text{abs}_\nu = dI^\text{abs}_\nu dA \cos \theta \, d\omega \, d\nu \, dt = -\kappa_\nu \, I_\nu \, dA \cos \theta \, d\omega \, dt \, d\nu \, ds \]

\[ dE^\text{em}_\nu = dI^\text{em}_\nu dA \cos \theta \, d\omega \, d\nu \, dt = \epsilon_\nu \, dA \cos \theta \, d\omega \, d\nu \, dt \, ds \]

\[ dE^\text{abs}_\nu + dE^\text{em}_\nu = (dI^\text{abs}_\nu + dI^\text{em}_\nu) \, dA \cos \theta \, d\omega \, d\nu \, dt = (-\kappa_\nu \, I_\nu + \epsilon_\nu) \, dA \cos \theta \, d\omega \, d\nu \, dt \, ds \]

\[ dI_\nu = dI^\text{abs}_\nu + dI^\text{em}_\nu = (-\kappa_\nu \, I_\nu + \epsilon_\nu) \, ds \]

\[ \frac{dI_\nu}{ds} = -\kappa_\nu \, I_\nu + \epsilon_\nu \]

differential equation describing the flow of radiation through matter
The equation of radiative transfer

Plane-parallel symmetry

\[ dx = \cos \theta \, ds = \mu \, ds \]

\[ \frac{d}{ds} = \mu \frac{d}{dx} \]

\[ \mu \frac{dI_\nu(\mu, x)}{dx} = -\kappa_\nu I_\nu(\mu, x) + \epsilon_\nu \]
The equation of radiative transfer

Spherical symmetry

angle \( \theta \) between ray and radial direction is not constant
The equation of radiative transfer

Optical depth and source function

In plane-parallel symmetry:

\[ \mu \frac{dI_\nu(\mu, x)}{dx} = -\kappa_\nu(x) I_\nu(\mu, x) + \epsilon_\nu(x) \]

\[ \mu \frac{dI_\nu(\mu, \tau_\nu)}{d\tau_\nu} = I_\nu(\mu, \tau_\nu) - S_\nu(\tau_\nu) \]

\[ S_\nu = \frac{\epsilon_\nu}{\kappa_\nu} \]

\[ \kappa_\nu = \frac{d\tau_\nu}{ds} \approx \frac{\Delta \tau_\nu}{\Delta s} \approx \frac{1}{\bar{s}} \]

\[ S_\nu = \frac{\epsilon_\nu}{\kappa_\nu} \approx \epsilon_\nu \cdot \bar{s} \]

Optical depth increasing towards interior:

\[ -\kappa_\nu \, dx = d\tau_\nu \]

\[ \tau_\nu = -\int_{R_i}^{x} \kappa_\nu \, dx \]

Source function

*Observed emerging intensity* \( I_\nu(\cos \theta, \tau_\nu = 0) \) depends on \( \mu = \cos \theta \), \( \tau_\nu(R_i) \) and \( S_\nu \)

The physics of \( S_\nu \) is crucial for radiative transfer.

\( \bar{s} \approx 1 \) corresponds to free mean path of photons

Source function \( S_\nu \) corresponds to intensity emitted over the free mean path of photons
The equation of radiative transfer

Source function: simple cases

a. LTE (thermal absorption/emission)

\[ S_\nu = \frac{\varepsilon_\nu}{\kappa_\nu} = B_\nu(T) \]

Kirchhoff's law: photons are absorbed and re-emitted at the local temperature \( T \)

Knowledge of \( T \) stratification \( T = T(x) \) or \( T(\tau) \) → solution of transfer equation \( I_\nu(\mu, \tau_\nu) \)
The equation of radiative transfer

Source function: simple cases

b. coherent isotropic scattering (e.g. Thomson scattering)

d. the absorption process is characterized by the scattering coefficient \( \sigma_\nu \), analogous to \( \kappa_\nu \):

\[
dI_\nu = -\sigma_\nu I_\nu ds
\]

and at each frequency \( \nu \):

\[
dE_{\nu}^{em} = dE_{\nu}^{abs} = \int \sigma_\nu I_\nu d\omega
\]

\[
\frac{\varepsilon_\nu^{sc}}{4\pi} \int d\omega = \sigma_\nu \int I_\nu d\omega
\]

\[
\frac{\varepsilon_\nu^{sc}}{\sigma_\nu} = \frac{1}{4\pi} \int I_\nu d\omega
\]

\[
S_\nu = J_\nu
\]

completely dependent on radiation field

not dependent on temperature \( T \)
The equation of radiative transfer

Source function: simple cases

c. mixed case

\[ S_\nu = \frac{\epsilon_\nu + \epsilon^{sc}_\nu}{\kappa_\nu + \sigma_\nu} = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu} \epsilon_\nu + \frac{\sigma_\nu}{\kappa_\nu + \sigma_\nu} \epsilon^{sc}_\nu \]

\[ S_\nu = \frac{\epsilon_\nu + \epsilon^{sc}_\nu}{\kappa_\nu + \sigma_\nu} = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu} B_\nu + \frac{\sigma_\nu}{\kappa_\nu + \sigma_\nu} J_\nu \]
Formal solution of the equation of radiative transfer

we want to solve the equation of RT with a known source function and in plane-parallel geometry

multiply by \( e^{-\tau/\mu} \) and integrate between \( \tau_1 \) (outside) and \( \tau_2 \) (> \( \tau_1 \), inside)

\[
\frac{d}{d\tau_\nu} \left( I_\nu \ e^{-\tau_\nu/\mu} \right) = -\frac{S_\nu \ e^{-\tau_\nu/\mu}}{\mu}
\]

check, whether this really yields transfer equation above

\[
\left[ I_\nu \ e^{-\tau_\nu/\mu} \right]_{\tau_1}^{\tau_2} = - \int_{\tau_1}^{\tau_2} S_\nu \ e^{-\tau_\nu/\mu} \ \frac{dt_\nu}{\mu}
\]
Formal solution of the equation of radiative transfer

\[ \left[ I_\nu e^{-\frac{\tau_\nu}{\mu}} \right]_{\tau_1}^{\tau_2} = - \int_{\tau_1}^{\tau_2} S_\nu e^{-\frac{\tau_\nu}{\mu}} \frac{dt_\nu}{\mu} \]

Formal solution of the equation of radiation transfer

\[ I_\nu(\tau_1, \mu) = I_\nu(\tau_2, \mu) e^{-\frac{\tau_2-\tau_1}{\mu}} + \int_{\tau_1}^{\tau_2} S_\nu(t) e^{-\frac{t-\tau_1}{\mu}} \frac{dt}{\mu} \]

intensity originating at \( \tau_2 \) decreased by exponential factor to \( \tau_1 \)

contribution to the intensity by emission along the path from \( \tau_2 \) to \( \tau_1 \)
(at each point decreased by the exponential factor)

Formal solution! actual solution can be complex, since \( S_\nu \) can depend on \( I_\nu \)
Boundary conditions

solution of RT equation requires boundary conditions, which are different for incoming and outgoing radiation

a. incoming radiation: $\mu < 0$ at $\tau_2 = 0$

usually we can neglect irradiation from outside: $I_\nu(\tau_2 = 0, \mu < 0) = 0$

$$I_\nu(\tau_1, \mu) = I_\nu(\tau_2, \mu) e^{-\frac{\tau_2 - \tau_1}{\mu}} + \int_{\tau_1}^{\tau_2} S_\nu(t) e^{-\frac{t-\tau_1}{\mu}} \frac{dt}{\mu}$$
b. outgoing radiation: $\mu > 0$ at $\tau_2 = \tau_{\text{max}} \to \infty$

We have either

$$I_\nu(\tau_{\text{max}}, \mu) = I_\nu^+(\mu)$$

finite slab or shell

or

$$\lim_{\tau \to \infty} I_\nu(\tau, \mu) e^{-\tau/\mu} = 0$$

semi-infinite case (planar or spherical)

$I_\nu$ increases less rapidly than the exponential

$$I_\nu^{\text{out}}(\tau_\nu, \mu) = \int_{\tau_\nu}^{\infty} S_\nu(t) e^{-t-\tau_\nu/\mu} \frac{dt}{\mu}$$

and at a given position $\tau_\nu$ in the atmosphere:

$$I_\nu(\tau_\nu) = I_\nu^{\text{out}}(\tau_\nu) + I_\nu^{\text{in}}(\tau_\nu)$$
Emergent intensity

from the latter $\rightarrow$ emergent intensity

$\tau_\nu = 0, \mu > 0$

$$I_\nu(0, \mu) = \int_0^\infty S_\nu(t) e^{-\frac{t}{\mu}} \frac{dt}{\mu}$$

intensity observed is a weighted average of the source function along the line of sight. The contribution to the emerging intensity comes mostly from each depths with $\tau/\mu < 1$. 
Emergent intensity

suppose that $S_\nu$ is linear in $\tau_\nu$ (Taylor expansion around $\tau_\nu = 0$):

$$S_\nu(\tau_\nu) = S_{0\nu} + S_{1\nu}\tau_\nu$$

$$I_\nu(0, \mu) = \int_0^\infty (S_{0\nu} + S_{1\nu}t)e^{-\frac{t}{\mu}} \frac{dt}{\mu} = S_{0\nu} + S_{1\nu}\mu$$

we see the source function at location $\tau_\nu = \mu$

the emergent intensity corresponds to the source function at $\tau_\nu = 1$ along the line of sight

\[ \int x e^{-x} dx = -(1 + x) e^{-x} \]
Emergent intensity

\[ \mu = 1 \text{ (normal direction):} \]
\[ I_\nu(0, 1) = S_\nu(\tau_\nu = 1) \]

\[ \mu = 0.5 \text{ (slanted direction):} \]
\[ I_\nu(0, 0.5) = S_\nu(\tau_\nu = 0.5) \]

in both cases: \( \Delta \tau/\mu \approx 1 \)

**spectral lines:** compared to continuum \( \tau_\nu/\mu = 1 \) is reached at higher layer in the atmosphere

\[ \Rightarrow S_\nu^{\text{line}} < S_\nu^{\text{cont}} \]

\[ \Rightarrow \text{a dip is created in the spectrum} \]
Line formation

simplify: \( \mu = 1, \tau_1 = 0 \) (emergent intensity), \( \tau_2 = \tau \)

\( S_\nu \) independent of location

\[
I_\nu(0) = I_\nu(\tau_\nu) e^{-\tau_\nu} + S_\nu \int_0^{\tau_\nu} e^{-t} dt = I_\nu(\tau_\nu) e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu})
\]
Line formation

Optically thick object:

\[ I_\nu(0) = I_\nu(\tau_\nu) e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu}) = S_\nu \]

\[ \tau_\nu \to \infty \]

Optically thin object:

\[ I_\nu(0) = I_\nu(\tau_\nu) + [S_\nu - I_\nu(\tau_\nu)] \tau_\nu \]

\[ \exp(-\tau_\nu) \approx 1 - \tau_\nu \]
\[ I_\nu = \tau_\nu \ k_\nu \ ds_\nu \ S_\nu \]

e.g. HII region, solar corona

e.g. stellar absorption spectrum (temperature decreasing outwards)

e.g. stellar spectrum with temperature increasing outwards (e.g. Sun in the UV)

\[ I_\nu(0) = I_\nu(\tau_\nu) + [S_\nu - I_\nu(\tau_\nu)] \tau_\nu \]

From Rutten's web notes
Line formation example: solar corona

\[ I_\nu = \tau_\nu S_\nu \]
The diffusion approximation

At large optical depth in stellar atmosphere photons are local: \( S_\nu \to B_\nu \)

Expand \( S_\nu (= B_\nu) \) as a power-series:

\[
S_\nu(t) = \sum_{n=0}^{\infty} \frac{d^n B_\nu}{d\tau_\nu^n} (t - \tau_\nu)^n / n!
\]

In the diffusion approximation \((\tau_\nu \gg 1)\) we retain only first order terms:

\[
B_\nu(t) = B_\nu(\tau_\nu) + \frac{dB_\nu}{d\tau_\nu}(t - \tau_\nu)
\]

\[
I^\text{out}_\nu(\tau_\nu, \mu) = \int_{\tau_\nu}^{\infty} [B_\nu(\tau_\nu) + \frac{dB_\nu}{d\tau_\nu}(t - \tau_\nu)] e^{-(t-\tau_\nu)/\mu} \frac{dt}{\mu}
\]
The diffusion approximation

Substituting: \[ t \rightarrow u = \frac{t - \tau}{\mu} \rightarrow dt = \mu\, du \]

\[
I^{\text{out}}_\nu (\tau, \mu) = \int_{\tau}^{\infty} \left[ B_\nu (\tau) + \frac{dB_\nu}{d\tau} (t - \tau) \right] e^{-\left(t - \tau\right)/\mu} \, dt
\]

\[
I^{\text{in}}_\nu (\tau, \mu) = - \int_{0}^{\tau/\mu} \left[ B_\nu (\tau) + \frac{dB_\nu}{d\tau} \, \mu u \right] e^{-u} \, du
\]

At \( \tau = 0 \) we obtain the Eddington-Barbier relation for the observed emergent intensity.

It is given by the Planck-function and its gradient at \( \tau = 0 \).

It depends linearly on \( \mu = \cos \theta \).
Solar limb darkening

\[
\frac{I_\nu(0,\mu)}{I_\nu(0,1)} = \frac{B_\nu(0) + \frac{dB_\nu}{d\tau_\nu} \mu}{B_\nu(0) + \frac{dB_\nu}{d\tau_\nu}}
\]

from the intensity measurements \(\rightarrow B_\nu(0), \frac{dB_\nu}{d\tau_\nu}\)

\[
B_\nu(t) = B_\nu(0) + \frac{dB_\nu}{d\tau_\nu} t = a + b \cdot t = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT(t)} - 1}
\]

\(T(t):\) empirical temperature stratification of solar photosphere
Solar limb darkening

...and also giant planets
Solar limb darkening: temperature stratification

\[ I_{\nu}(0, \mu) = \int_{0}^{\infty} S_{\nu}(t) e^{-\frac{t}{\mu}} \frac{dt}{\mu} \]

exponential extinction varies as \(-\tau_{\nu}/\cos \theta\)

From \(S_{\nu} = a + b \tau_{\nu}\):

\[ I_{\nu}(0, \cos \theta) = a_{\nu} + b_{\nu} \cos \theta \]

\[ I_{\nu}(0, \mu) = S_{\nu}(\tau_{\nu} = \mu) \rightarrow S_{\nu} \]

R. Rutten, web notes

Unsoeld, 68
In the diffusion approximation we had:

\[ B_\nu(t) = B_\nu(\tau_\nu) + \frac{dB_\nu}{d\tau_\nu}(t - \tau_\nu) \]

\[ I_\nu^{\text{out}}(\tau_\nu, \mu) = B_\nu(\tau_\nu) + \mu \frac{dB_\nu}{d\tau_\nu} \]

\[ 0 < \mu < 1 \]

\[ I_\nu^{\text{in}}(\tau_\nu, \mu) = - \int_{0}^{\tau_\nu/\mu} \left[ B_\nu(\tau_\nu) + \frac{dB_\nu}{d\tau_\nu} \mu u \right] e^{-u} du \]

\[ -1 < \mu < 0 \]

\[ \tau \gg 1 \quad I_\nu^{-}(\tau_\nu, \mu) = B_\nu(\tau_\nu) + \mu \frac{dB_\nu}{d\tau_\nu} \]

we want to obtain an approximation for the radiation field – both inward and outward radiation - at large optical depth

\[ \rightarrow \text{stellar interior, inner boundary of atmosphere} \]
Eddington approximation

\[ J_\nu = \frac{1}{2} \int_{-1}^{1} I_\nu \, d\mu = B_\nu(\tau_\nu) \]

\[ H_\nu = \frac{F_\nu}{4} = \frac{1}{2} \int_{-1}^{1} \mu \, I_\nu \, d\mu = \frac{1}{3} \frac{dB_\nu}{d\tau_\nu} = -\frac{1}{3} \frac{1}{\kappa_\nu} \frac{dB_\nu}{dx} = -\frac{1}{3\kappa_\nu} \frac{dB_\nu}{dT} \frac{dT}{dx} \]

\[ K_\nu = \frac{1}{2} \int_{-1}^{1} \mu^2 \, I_\nu \, d\mu = \frac{1}{3} B_\nu(\tau_\nu) \]

With this approximation for \( I_\nu \) we can calculate the angle averaged momenta of the intensity

→ simple approximation for photon flux and a relationship between mean intensity \( J_\nu \) and \( K_\nu \)

→ very important for analytical estimates

flux \( F_\nu \sim dT/dx \)

diffusion: flux \( \sim \) gradient (e.g. heat conduction)
Schwarzschild-Milne equations

After the previous approximations, we now want to calculate exact solutions for the radiative momenta $J_\nu$, $H_\nu$, $K_\nu$. Those are important to calculate spectra and atmospheric structure.

\[
J_\nu = \frac{1}{2} \left[ \int_{-1}^{1} I_\nu d\mu + \frac{1}{2} \int_{0}^{1} I_\nu^{out} d\mu + \frac{1}{2} \int_{-1}^{0} I_\nu^{in} d\mu \right]
\]

\[
I_\nu^{out}(\tau_\nu, \mu) = \int_{\tau_\nu}^{\infty} S_\nu(t) e^{-\frac{t - \tau_\nu}{\mu}} \frac{dt}{\mu}
\]

\[
I_\nu^{in}(\tau_\nu, \mu) = \int_{0}^{\tau_\nu} S_\nu(t) e^{-\frac{t - \tau_\nu}{\mu}} \frac{dt}{\mu}
\]

\[
J_\nu = \frac{1}{2} \left[ \int_{0}^{\infty} \int_{\tau_\nu}^{\infty} S_\nu(t) e^{-(t - \tau_\nu) / \mu} \frac{dt}{\mu} d\mu - \int_{-1}^{0} \int_{0}^{\tau_\nu} S_\nu(t) e^{-(t - \tau_\nu) / \mu} \frac{dt}{\mu} d\mu \right]
\]

Substitute $w = \frac{1}{\mu} \Rightarrow \frac{dw}{w} = -\frac{1}{\mu} d\mu$

$w = -\frac{1}{\mu} \Rightarrow \frac{dw}{w} = -\frac{1}{\mu} d\mu$

\[
J_\nu = \frac{1}{2} \left[ \int_{1}^{\infty} \int_{\tau_\nu}^{\infty} S_\nu(t) e^{-(t - \tau_\nu)w} \frac{dt}{w} dw + \int_{1}^{\infty} \int_{0}^{\tau_\nu} S_\nu(t) e^{-(\tau_\nu - t)w} \frac{dt}{w} dw \right]
\]
Schwarzschild-Milne equations

\[ J_\nu = \frac{1}{2} \left[ \int_{-\infty}^{\infty} S_\nu(t) \int_{-\infty}^{\infty} e^{-(t-\tau_\nu)w} \frac{dw}{w} dt + \int_{0}^{\infty} S_\nu(t) \int_{-\infty}^{\infty} e^{-(\tau_\nu-t)w} \frac{dw}{w} dt \right] > 0 \]

\[ J_\nu = \frac{1}{2} \int_{0}^{\infty} S_\nu(t) \int_{1}^{\infty} e^{-w|t-\tau_\nu|} \frac{dw}{w} dt = \frac{1}{2} \int_{0}^{\infty} S_\nu(t) E_1(|t-\tau_\nu|) dt \]

Schwarzschild’s equation
Schwarzschild-Milne equations

where

\[ E_1(t) = \int_1^\infty e^{-tx} \frac{dx}{x} = \int_t^\infty e^{-x} \frac{dx}{x} \]

is the first exponential integral (singularity at t=0)

Exponential integrals

\[ E_n(t) = t^{n-1} \int_t^\infty x^{-n} e^{-x} dx \]

\[ E_n(0) = 1/(n-1), \quad E_n(t \to \infty) = e^{-t}/t \to 0 \]

\[ \frac{dE_n}{dt} = -E_{n-1}, \quad \int E_n(t) = -E_{n+1}(t) \]

\[ E_1(0) = \infty \quad E_2(0) = 1 \quad E_3(0) = 1/2 \quad E_n(\infty) = 0 \]
Schwarzschild-Milne equations

Introducing the $\Lambda$ operator:

$$\Lambda_{\tau_\nu}[f(t)] = \frac{1}{2} \int_0^\infty f(t) E_1(|t - \tau_\nu|) \, dt$$

Similarly for the other 2 moments of Intensity:

$$H_\nu(\tau_\nu) = \frac{1}{2} \int_{\tau_\nu}^\infty S_\nu(t) E_2(t - \tau_\nu) \, dt - \frac{1}{2} \int_0^{\tau_\nu} S_\nu(t) E_2(\tau_\nu - t) \, dt = \Phi_{\tau_\nu}(S_\nu(t))$$

$$K_\nu(\tau_\nu) = \frac{1}{2} \int_0^\infty S_\nu(t) E_3(|t - \tau_\nu|) \, dt = X_{\tau_\nu}(S_\nu(t))$$

$J_\nu$, $H_\nu$ and $K_\nu$ are all depth-weighted means of $S_\nu$.
Schwarzschild-Milne equations

the 3 moments of Intensity:

\[ J_\nu(\tau_\nu) = \frac{1}{2} \int_0^{\infty} S_\nu(t) E_1(\mid t - \tau_\nu \mid) dt = \Lambda_{\tau_\nu}(S_\nu(t)) \]

\[ H_\nu(\tau_\nu) = \frac{1}{2} \int_{\tau_\nu}^{\infty} S_\nu(t) E_2(t - \tau_\nu) dt - \frac{1}{2} \int_0^{\tau_\nu} S_\nu(t) E_2(\tau_\nu - t) dt = \Phi_{\tau_\nu}(S_\nu(t)) \]

\[ K_\nu(\tau_\nu) = \frac{1}{2} \int_0^{\infty} S_\nu(t) E_3(\mid t - \tau_\nu \mid) dt = X_{\tau_\nu}(S_\nu(t)) \]

\( J_\nu, H_\nu \) and \( K_\nu \) are all depth-weighted means of \( S_\nu \)

→ the strongest contribution comes from the depth, where the argument of the exponential integrals is zero, i.e. \( t = \tau_\nu \)
The temperature-optical depth relation

Radiative equilibrium

The condition of radiative equilibrium (expressing conservation of energy) requires that the flux at any given depth remains constant:

\[ \mathcal{F}(r) = \pi F = \int_0^\infty \int \frac{I_\nu}{4\pi} \cos \theta \, d\omega \, d\nu = \pi \int_0^\infty F_\nu \, d\nu = 4\pi \int_0^\infty H_\nu \, d\nu \]

\[ 4\pi r^2 \mathcal{F}(r) = 4\pi r^2 \cdot 4\pi \int_0^\infty H_\nu \, d\nu = \text{const} = L \]

In plane-parallel geometry \( r \approx R = \text{const} \)

\[ 4\pi \int_0^\infty H_\nu \, d\nu = \text{const} \]

and in analogy to the black body radiation, from the Stefan-Boltzmann law we define the effective temperature:

\[ 4\pi \int_0^\infty H_\nu \, d\nu = \sigma T_{\text{eff}}^4 \]
The **effective temperature**

The effective temperature is defined by:

\[ 4\pi \int_0^\infty H_\nu \, d\nu = \sigma T_{\text{eff}}^4 \]

It characterizes the total radiative flux transported through the atmosphere.

It can be regarded as an average of the temperature over depth in the atmosphere.

A blackbody radiating the same amount of total energy would have a temperature \( T = T_{\text{eff}} \).
Let us now combine the condition of radiative equilibrium with the equation of radiative transfer in plane-parallel geometry:

\[ \mu \frac{dI_\nu}{dx} = - (\kappa_\nu + \sigma_\nu)(I_\nu - S_\nu) \]

\[ \frac{1}{2} \int_{-1}^{1} \mu \frac{dI_\nu}{dx} \, d\mu = - \frac{1}{2} \int_{-1}^{1} (\kappa_\nu + \sigma_\nu)(I_\nu - S_\nu) \, d\mu \]

\[ \frac{d}{dx} \left[ \frac{1}{2} \int_{-1}^{1} \mu I_\nu \, d\mu \right] = - (\kappa_\nu + \sigma_\nu)(J_\nu - S_\nu) \]

\[ H_\nu \]
Radiative equilibrium

Integrate over frequency:

\[
\frac{d}{dx} \int_0^\infty H_\nu \, d\nu = - \int_0^\infty (\kappa_\nu + \sigma_\nu) (J_\nu - S_\nu) \, d\nu
\]

\[
\text{const}
\]

\[
\int_0^\infty (\kappa_\nu + \sigma_\nu) (J_\nu - S_\nu) \, d\nu = 0
\]

at each depth:

\[
\int_0^\infty \kappa_\nu [J_\nu - B_\nu(T)] \, d\nu = 0
\]

in addition:

\[
4\pi \int_0^\infty H_\nu \, d\nu = \sigma T_{\text{eff}}^4
\]

\[
\text{T(x) or T(τ)}
\]
Radiative equilibrium

\[ \int_0^\infty \kappa_\nu \left[ J_\nu - B_\nu(T) \right] d\nu = 0 \]

\[ 4\pi \int_0^\infty H_\nu \ d\nu = \sigma T_{\text{eff}}^4 \]

The temperature \( T(r) \) at every depth has to assume the value for which the left integral over all frequencies becomes zero.

\[ \Rightarrow \] This determines the local temperature.
Iterative method for calculation of a stellar atmosphere:

the major parameters are $T_{\text{eff}}$ and $g$

- a. hydrostatic equilibrium
  $$\frac{dP}{dx} = -g\rho(x)$$

- b. equation of radiation transfer
  $$\mu \frac{dI_\nu}{dx} = -(\kappa_\nu + \sigma_\nu)(I_\nu - S_\nu)$$

- c. radiative equilibrium
  $$\int_0^\infty \kappa_\nu [J_\nu - B_\nu(T)] d\nu = 0$$

- d. flux conservation
  $$4\pi \int_0^\infty H_\nu d\nu = \sigma T_{\text{eff}}^4$$

- e. equation of state
  $$P = \rho \frac{k T}{\mu m_H}$$
Grey atmosphere - an approximation for the temperature structure

We derive a simple analytical approximation for the temperature structure. We assume that we can approximate the radiative equilibrium integral by using a frequency-averaged absorption coefficient, which we can put in front of the integral.

\[
\int_0^\infty \kappa_\nu [J_\nu - B_\nu(T)] \, d\nu = 0 \quad \Rightarrow \quad \bar{\kappa} \int_0^\infty [J_\nu - B_\nu(T)] \, d\nu = 0
\]

With:

\[
J = \int_0^\infty J_\nu \, d\nu \quad H = \int_0^\infty H_\nu \, d\nu \quad K = \int_0^\infty K_\nu \, d\nu \quad B = \int_0^\infty B_\nu \, d\nu = \frac{\sigma T^4}{\pi}
\]

\[
J = B \\
4\pi H = \sigma T_{\text{eff}}^4
\]
Grey atmosphere

We then assume LTE: \( S = B \).

From

\[
J_\nu(\tau_\nu) = \Lambda_{\tau_\nu}[S_\nu(t)] = \frac{1}{2} \int_0^\infty S_\nu(t) E_1(|t - \tau_\nu|) \, dt
\]

and a similar expression for frequency-integrated quantities

\[
J(\bar{\tau}) = \Lambda_{\bar{\tau}}[S(t)], \quad d\bar{\tau} = \bar{\kappa} \, dx
\]

and with the approximations \( S = B, \, B = J \):

\[
J(\bar{\tau}) = \Lambda_{\bar{\tau}}[J(t)] = \frac{1}{2} \int_0^\infty J(t) E_1(|t - \bar{\tau}|) \, dt
\]

Milne’s equation

!!! this is an integral equation for \( J(\tau) \) !!!
The exact solution of the Hopf integral equation

Milne’s equation \( J(\tau) = \Lambda_{\tau}[J(t)] \) \( \rightarrow \) exact solution (see Mihalas, “Stellar Atmospheres”)

\[
J(\tau) = \text{const.} \ [ \tau + q(\tau)], \quad \text{with } q(\tau) \text{ monotonic}
\]

\[
\frac{1}{\sqrt{3}} = 0.577 = q(0) \leq q(\bar{\tau}) \leq q(\infty) = 0.710
\]

Radiative equilibrium - grey approximation

\[
J(\tau) = B(\tau) = \sigma/\pi \ T^4(\tau) = \text{const.} \ [ \tau + q(\tau)]
\]

with boundary conditions \( \rightarrow \)

\[
T^4(\tau) = \frac{3}{4} \ T^4_{\text{eff}} [\tau + q(\tau)]
\]
A simple approximation for $T(\tau)$

0\textsuperscript{th} moment of equation of transfer (integrate both sides in $d\mu$ from -1 to 1)

$$\mu \frac{dI}{dx} = -\bar{\kappa}(I - B) \quad \rightarrow \quad \frac{dH}{d\tau} = J - B = 0 \quad (J = B) \quad \rightarrow \quad H = \text{const} = \frac{\sigma T_{\text{eff}}^4}{4\pi}$$

1\textsuperscript{st} moment of equation of transfer (integrate both sides in $\mu d\mu$ from -1 to 1)

$$\mu \frac{dI}{dx} = -\bar{\kappa}(I - B) \quad \rightarrow \quad \frac{dK}{d\tau} = H = \frac{\sigma T_{\text{eff}}^4}{4\pi} \quad \rightarrow \quad K(\bar{\tau}) = H\bar{\tau} + \text{constant}$$

From Eddington’s approximation at large depth: $K = 1/3 \, J$

$$J(\bar{\tau}) = 3H(\bar{\tau} + c) = \frac{\sigma T_{\text{eff}}^4}{\pi}$$
Grey atmosphere – temperature distribution

\[ J(\tau) = 3H(\tau + c) = \frac{\sigma T^4}{\pi} \]

\[ T^4(\tau) = \frac{3\pi H}{\sigma} (\tau + c) \quad \quad H = \frac{\sigma}{4\pi} T_{\text{eff}}^4 \]

\[ T^4(\tau) = \frac{3}{4} T_{\text{eff}}^4 (\tau + c) \quad \quad T^4 \text{ is linear in } \tau \]

**Estimation of c**

\[ H(\tau = 0) = \frac{1}{2} \int_0^\infty J(t)E_2(t)\,dt = \frac{1}{2} 3H \int_0^\infty (t + c)E_2(t)\,dt \]

\[ H_{\tau}(\tau = 0) = \frac{1}{2} 3H \left[ \int_0^\infty tE_2(t)\,dt + c \int_0^\infty E_2(t)\,dt \right] \]

\[ \int_0^\infty t^s E_n(t)\,dt = \frac{s!}{s + n} \]
Grey atmosphere – Hopf function

\[ H(0) = H = \frac{1}{2} H(1 + \frac{3}{2} c) \rightarrow c = \frac{2}{3} \]

\[ T^4(\bar{\tau}) = \frac{3}{4} T_{\text{eff}}^4 \left( \bar{\tau} + \frac{2}{3} \right) \]

Based on approximation \( K/J = 1/3 \)

\[ T = T_{\text{eff}} \text{ at } \tau = 2/3, \ T(0) = 0.84 \ T_{\text{eff}} \]

Remember: More in general \( J \) is obtained from

\[ J(\bar{\tau}) = \Lambda_{\bar{\tau}} [S(t)] \]

\[ T^4(\bar{\tau}) = \frac{3}{4} T_{\text{eff}}^4 \left( \bar{\tau} + q(\bar{\tau}) \right) \]

\( q(\bar{\tau}) \) : Hopf function

Once Hopf function is specified \( \rightarrow \) solution of the grey atmosphere (temperature distribution)

\[ \frac{1}{\sqrt{3}} = 0.577 = q(0) \leq q(\bar{\tau}) \leq q(\infty) = 0.710 \]
Selection of the appropriate \( \kappa_\nu \Rightarrow \bar{\kappa} \)

In the grey case we define a ‘suitable’ mean opacity (absorption coefficient).

\[
\kappa_\nu \Rightarrow \bar{\kappa} \quad I = \int_0^\infty I_\nu \, d\nu \quad J = \int_0^\infty J_\nu \, d\nu \quad \ldots
\]

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<thead>
<tr>
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<th>non-grey</th>
<th>grey</th>
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<td>( \mu \frac{dI_\nu}{dx} = -\kappa_\nu (I_\nu - S_\nu) )</td>
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Selection of the appropriate $\kappa_\nu \Rightarrow \bar{\kappa}$

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For each equation there is one opacity average that fits “grey equations”, however, all averages are different. Which one to select?

$\Rightarrow$ For flux constant models with $H(\tau) = \text{const.}$ 2nd moment equation is relevant $\Rightarrow$
Mean opacities: flux-weighted

1st possibility: Flux-weighted mean

\[ \bar{\kappa} = \frac{\int_0^\infty \kappa_\nu H_\nu \, d\nu}{H} \]

allows the preservation of the K-integral (radiation pressure)

Problem: \( H_\nu \) not known a priory (requires iteration of model atmospheres)
Mean opacities: Rosseland

2nd possibility: Rosseland mean

to obtain correct integrated energy flux and use local $T$

\[
\int_0^\infty \frac{1}{\kappa_\nu} \frac{dK_\nu}{dx} \, d\nu = -H \Rightarrow (grey) \Rightarrow \frac{1}{\bar{\kappa}} \frac{dK}{dx} = -H
\]

\[
\frac{1}{\bar{\kappa}} = \frac{ \int_0^\infty \frac{1}{\kappa_\nu} \frac{dK_\nu}{dx} \, d\nu}{\frac{dK}{dx}}
\]

\[K_\nu \rightarrow \frac{1}{3} J_\nu, \quad J_\nu \rightarrow B_\nu \text{ as } \tau \rightarrow \infty\]

\[
\frac{dK_\nu}{dx} \rightarrow \frac{1}{3} \frac{dB_\nu}{dx} = \frac{1}{3} \frac{dB_\nu}{dT} \frac{dT}{dx}
\]

\[
\frac{1}{\bar{\kappa}_{Ross}} = \frac{ \int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu(T)}{dT} \, d\nu}{\int_0^\infty \frac{dB_\nu(T)}{dT} \, d\nu}
\]

large weight for low-opacity (more transparent to radiation) regions
Mean opacities: Rosseland

at large $\tau$ the T structure is accurately given by

$$ T^4 = \frac{3}{4} T_{\text{eff}}^4 \left[ \tau_{\text{Ross}} + q(\tau_{\text{Ross}}) \right] $$

Rosseland opacities used in stellar interiors

For stellar atmospheres Rosseland opacities allow us to obtain initial approximate values for the Temperature stratification (used for further iterations).
$T^4$ vs. $\tau$

Grey: $q(\tau) = \frac{2}{3}$

Non-grey numerical
\[ T \text{ vs. } \log(\tau) \]

grey: \( q(\tau) = \text{exact} \)

grey: \( q(\tau) = 2/3 \)

non-grey numerical
Iterative method for calculation of a stellar atmosphere:

the major parameters are $T_{\text{eff}}$ and $g$

\[ T(x), \kappa_\nu(x), B_\nu[T(x)], P(x), \rho(x) \]

\[ J_\nu(x), H_\nu(x) \]

\[ \int_0^\infty \kappa_\nu(J_\nu - B_\nu) \, d\nu = 0? \]

\[ 4\pi \int_0^\infty H_\nu \, d\nu = \sigma T_{\text{eff}}^4? \]

\[ \Delta T(x), \Delta \kappa_\nu(x), \Delta B_\nu[T(x)], \Delta \rho(x) \]

\[ \frac{dP}{dx} = -g\rho(x) \]

b. equation of radiation transfer

\[ \mu \frac{dI_\nu}{dx} = - (\kappa_\nu + \sigma_\nu)(I_\nu - S_\nu) \]

c. radiative equilibrium

\[ \int_0^\infty \kappa_\nu [J_\nu - B_\nu(T)] \, d\nu = 0 \]

d. flux conservation

\[ 4\pi \int_0^\infty H_\nu \, d\nu = \sigma T_{\text{eff}}^4 \]

e. equation of state

\[ P = \frac{\rho k T}{\mu m_H} \]