IV. Line Formation in Expanding Atmospheres

1. Doppler - Effect

Stellar winds $\rightarrow$ velocity fields $\rightarrow$ need to be accounted for in radiative transfer

$$\frac{dI_\nu}{ds} = -\kappa_\nu(s)I_\nu + \epsilon_\nu(s)$$

$$\frac{dI_\nu}{ds} = -\kappa_\nu(s, \vec{V})I_\nu + \epsilon_\nu(s, \vec{V})$$

Velocity fields affect $\kappa_\nu$, $\epsilon_\nu$ through Doppler - Effect

$$\nu' = \nu \gamma (1 - \beta \cos \theta)$$

$$\gamma = (1 - \beta)^{1/2}$$

$$\beta = \frac{|\vec{V}|}{c}$$

$\nu'$ = frequency of photon in moving system
$\nu$ = observer
$\theta$ = angle between photon direction and $\vec{V}(\vec{r})$

Relativistic formula including “transversal” effect at $\theta=90^\circ$
stellar winds are non-relativistic with $V/c \ll 1$

\[\nu' = \nu (1 - \beta \cos \theta)\]
\[\nu' = \nu \left(1 - \frac{|\vec{V}|}{c} \cos \theta\right)\]
\[\nu' = \nu \left(1 - \frac{1}{c} \vec{V} \cdot \vec{n}\right)\]

Atom moving with $\vec{V}$ "sees" photon with $\nu'$

non-moving observer "sees" photon with $\nu$

non-moving photosphere emits photon with $\nu$
2. The line absorption coefficient

line absorption coefficient in static case with \( \vec{V} = 0 \)

\[
\kappa_{\nu}^{\text{line}} = \frac{\pi e^2}{m c} f_{lu} \{ n_l(s) - \frac{g_l}{g_u} n_u(s) \} \phi(\nu)
\]

- “classical” cross section
- oscillator strength
- occupation numbers \( l,u \)
- line broadening

\[ \phi(\nu) \]

contains thermal line broadening
natural pressure

in winds → thermal broadening

thermal Doppler width

\[
\phi(\nu) = \frac{1}{\Delta \nu_D \sqrt{\pi}} e^{-\left(\frac{\nu - \nu_0}{\Delta \nu_D}\right)^2}
\]

\[
\Delta \nu_D = \nu_0 \frac{v_{th}}{c} = \nu_0 \frac{2kT}{m_{ion} c^2}
\]
Introducing the “dimensionless frequency” $x$

$$x = \frac{\nu - \nu_0}{\Delta \nu_D}$$

and the line profile $\phi(x)$

$$\phi(x) = \frac{1}{\pi^{1/2}} e^{-x^2}$$

$$k^\text{line}_\nu(s) = k(s) \phi(x)$$

we obtain the static line absorption coefficient

$$k(s) = \frac{\pi e^2}{mc} f_{lu} \left\{ n_l - \frac{g_l}{g_u} n_u \right\} \frac{1}{\Delta \nu_D}$$

Note: the static line absorption coefficient is

(i) is isotropic (no dependence on angle $\theta$)

(ii) varies smoothly as a function of $s$ through $k(s)$

(iii) varies strongly as a function of $x$ through $\phi(x)$
hydrostatic atmosphere

smooth function of $s$

strong function of $\nu$

width determined by $\Delta \nu_D$

$\phi(x)$
Hydrodynamic atmosphere with $\vec{V} \neq 0$

Atom moving with $\vec{V}(s)$ sees shifted frequencies

$\nu \quad \rightarrow \quad \nu'$

$\nu' - \nu = -\nu_0 \mu \frac{V(s)}{c}$

$\mu = \cos \theta \quad V(s) = |\vec{V}(s)|$

Profile function changes from

$\phi(x) = \pi^{-1/2} e^{-x^2} = \pi^{-1/2} e^{-\left(\frac{\nu - \nu_0}{\Delta \nu_D}\right)^2}$

to

$\phi\left(x - \frac{\nu_0 \mu V(s)}{c \Delta \nu_D}\right) = \pi^{-1/2} e^{-\left(\frac{\nu - \nu_0}{\Delta \nu_D} - \frac{\nu_0}{c} \frac{\mu V(s)}{\Delta \nu_D}\right)^2}$
Introducing

\[ \vec{v}(r) = \frac{1}{v_{th}} \vec{V}(r) \]

velocity in units of thermal velocity

and using

\[ v_{th} = \frac{c}{\nu_0} \Delta \nu_D \]

we obtain the absorption coefficient in the hydrodynamic case

\[ \phi(x) \rightarrow \phi(x - \mu \nu(r)) = \pi^{-1/2} e^{-(x-\mu \nu)^2} \]

light ray passes through velocity field absorbed by atoms with different Doppler shifts

\[ \kappa_{\nu}^{line} = k(s) \phi(x - \mu \nu(s)) \]

\[ \phi(x - \mu \nu(s)) \rightarrow \text{anisotropic} \]

\[ \phi(x - \mu \nu(s)) \rightarrow \text{extreme depth dependent} \]
frequency of photon as seen by gas

at \( s = s_0 \): \( \nu' = \nu_0 \)  resonance absorption

projected velocity

at \( s = s_0 \): \( x = \mu \nu \) resonance absorption

\( \kappa_\nu \) has strong maximum

at \( s = s_0 \):

\[
\tau(s) = \int \kappa ds
\]

optical depth is step function

\( \tau_S \) optical thickness of interaction region

gеometrical width of interaction region
radiative line transfer with \( V(r) \gg v_{th} \)

photon - gas interaction restricted to thin interaction zone around the resonance condition defined by

\[
x = \mu \, v(r)
\]

or

\[
x = \vec{n} \cdot \vec{v}(r)
\]

Note: hydrostatic radiative line transfer

photons reach observer from layers with \( \tau \approx 1 \)

hydrodynamic radiative line transfer

photons come from interaction zone defined by resonance condition independent of value of \( \tau \)
3. Interaction surfaces in spherical symmetry

“(p,z) – geometry”

- all rays characterized by impact parameter p
- z coordinate along ray
- p, z, r all in units of stellar photospheric radius $R_*$

\[
\begin{align*}
r^2 &= p^2 + z^2 \\
z &= \mu \ r \\
p &= r \ (1 - \mu^2)^{1/2}
\end{align*}
\]
For each ray with frequency $x$ and impact parameter $p$ exists interaction point $z_0$ where $x = \mu v$

the set of all interaction points is $z_0(x, p)$ “interaction surface”

the shape of $z_0(x, p)$ depends on the velocity field $v(r)$ observed $v(r)$ for hot stars

$$v(r) = v_\infty \left(1 - \frac{1}{r}\right)\beta$$

$\beta = 0.5..4.0$

$$\frac{dv}{dr} \geq 0 \quad v \to v_\infty, \ r \gg 1$$
general shape of interaction surfaces
discussion of \( z_0(x, p) \)

a) line center frequency \( x=0 \)

resonance condition

\[
x = \mu \frac{z}{r} v(r)
\]

interaction surface

for \( p \leq 1 \): \( r = 1 \), since \( v(r) = 0 \)

\( p > 1 \): \( z = 0 \), since \( r > 1 \)

and \( v(r) > 0 \)
b) radial axis towards observer with $p = 0, x \neq 0$

$x = \mu v(r)$ but $p = 0 \rightarrow \mu = 1$

$p = r (1 - \mu^2)^{1/2}$

$x = v(r)$

$r^2 = p^2 + z^2$

$x = v_\infty \left(1 - \frac{1}{(p^2 + z_0^2)^{1/2}}\right)^\beta = v_\infty (1 - \frac{1}{z_0})^\beta$

$z_0, p=0 = \frac{1}{1 - \left(\frac{x}{v_\infty}\right)^{1/\beta}}$

for $x < x_{\text{max}} = v_\infty$

max. Doppler-shift
c) \( p \neq 0 \), but \( \frac{x}{v_\infty} \to 1 \)

\[
x/v_\infty \text{ approaching } 1 \quad \Rightarrow \quad z_0(x, p) \gg 1
\]

\[
x = \frac{z_0}{(p^2 + z_0^2)^{1/2}} \cdot v_\infty \left(1 - \frac{1}{(p^2 + z_0^2)^{1/2}}\right)^\beta
\]

\[
\left(\frac{x}{v_\infty}\right)^2 (p^2 + z_0^2) = z_0^2 \left(1 - \frac{1}{(p^2 + z_0^2)^{1/2}}\right)^{2\beta}
\]

\[
\approx z_0^2 \left(1 - \frac{2\beta}{(p^2 + z_0^2)^{1/2}}\right)
\]

solve for \( z_0^2 \)
\[ z_0^2(x, p) = \frac{(x/v_\infty)^2}{\left\{ 1 - \frac{x^2}{v_\infty^2} - \frac{2\beta}{(p^2 + z_0^2)^{1/2}} \right\}} p^2 \]

For \( p = 0 \) \( \Rightarrow \{\} = 0 \)

\[ z_0, p=0 = \frac{2\beta}{1 - \frac{x^2}{v_\infty^2}} = \frac{2}{1 + \frac{x}{v_\infty}} \frac{\beta}{1 - \frac{x}{v_\infty}} \]

\(|p| > 0 \Rightarrow \{\} > 0 \)

\[ \{\} = 2\beta \left( \frac{1}{z_0, p=0} - \frac{1}{(z_0^2 + p^2)^{1/2}} \right) \]

\(|p| >> 1 \Rightarrow \]

\[ (z_0^2 + p^2)^{1/2} \gg z_0, p=0 \]

\[ z_0(x, p) = \frac{x/v_\infty}{(1 - \frac{x^2}{v_\infty^2})^{1/2}} p \]

interaction surface linear with \( p \) at large \( p \)
general shape of interaction surfaces

detailed calculation by numerical solution of

\[ f(z_0, x, p, \beta) = 0 \]

\[ f = \frac{x}{v_\infty} - \frac{z_0}{r_0} \left(1 - \frac{1}{r_0}\right)^\beta \quad r_0 = (z_0^2 + p^2)^{1/2} \]
$\beta = 2$

$x/v_\infty = -0.9, -0.8, -0.7 \ldots +0.7, +0.8, +0.9$
$\beta = 1$

interaction surfaces

$x/v_\infty = -0.9, -0.8, -0.7 \ldots +0.7, +0.8, +0.9$
a very simple case: linear velocity fields

\[ v(r) = a \cdot r \]

resonance condition \[ x = \mu v(r) = \frac{z_0}{r} v(r) \]

\[ x = \frac{z_0}{r} a \cdot r = z_0 a \]

\[ z_0 = \frac{x}{a} \]

independent of p !!!

plane interaction surfaces

Supernovae have linear velocity fields
4. Optical thickness of interaction surface

Optical thickness of interaction surface determines how much is absorbed and re-emitted.

Calculate $\tau$ along ray with $p = \text{const.}$

$$\tau(x, p, z) = R_\star \int_{z_{\text{min}}}^{z} k[r(\tilde{z}, p)] \phi[x - \mu(\tilde{z}, p) v(r(\tilde{z}, p))] d\tilde{z}$$

Integrand has peak at interaction point $z_0$, where

$$x - \mu(z_0, p) v(r(z_0, p)) = 0$$

$\frac{\partial(\mu v)}{\partial z}$ large $\rightarrow$ peak narrow

$\rightarrow k(r) = k(r_0) = \text{const}$ over peak
substitute \( u = x - \mu v \) \[ d\tilde{z} = -\left\{ \frac{\partial(\mu v)}{\partial \tilde{z}} \right\}^{-1} du \]

\[
\tau(x, p, z) = R_\star \int u(x, p, z_{\text{min}}) k(r) \left\{ \frac{\partial(\mu v)}{\partial \tilde{z}} \right\}^{-1} \phi(u) du
\approx R_\star k(r(z_0, p)) \left\{ \frac{\partial(\mu v)}{\partial \tilde{z}} \right\}^{-1} \int u(z_{\text{min}}) \phi(u) du
\]

since \( u(z_{\text{min}}) \gg 1 \)

\[
\Phi(u) \approx \int_u^\infty \phi(\tilde{u}) d\tilde{u} = \pi^{-1/2} \int_u^\infty e^{-\tilde{u}^2} d\tilde{u} = \text{Erfc}(u)
\]

step function

\[
\Phi(u) = 0, \quad u > 0 \\
\Phi(u) = 1, \quad u \leq 0
\]
u as a function of z

\[ u = x - \mu \quad v = x - \frac{z}{(z^2 + p^2)^{1/2}} \quad v_{\infty} (1 - \frac{1}{r})^{\beta} \]

For \( z = z_{\text{min}} \rightarrow -\infty \rightarrow u \rightarrow u = x + v_{\infty} \)
\[ \tau(x, p, z) \approx \tau_s(p, z_0) \Phi[u(x, p, z)] \]

\[ \frac{\partial}{\partial z} \{\mu(z, p) \, v(r(z, p))\} = \mu \frac{\partial v}{\partial r} \frac{\partial r}{\partial z} + v \frac{\partial \mu}{\partial z} \]

\[ \frac{\partial}{\partial z} (\mu v) = \mu^2 \frac{\partial v}{\partial r} + \frac{v}{r} (1 - \mu^2) \]

\[ \tau_s = R_* k(r) \frac{\partial (\mu v)}{\partial z}^{-1} \quad p = \text{const.} \]

\[ p^2 + z^2 = r^2 \quad z = r \mu \]

\[ r \, dr = z \, dz \]

\[ \frac{\partial r}{\partial z} = \frac{z}{r} = \mu \]

\[ \frac{\partial \mu}{\partial z} = \frac{1}{r} - \frac{z}{r^2} \mu \]

\[ \tau_s = R_* \frac{k(r)}{Q(r, \mu)} = R_* \frac{k(r)}{\mu^2 \frac{\partial v}{\partial r} + (1 - \mu^2) \frac{v}{r}} \]
discussion:

$$\tau_s = R_* \frac{k(r)}{Q(r, \mu)} = R_* \frac{k(r)}{\mu^2 \frac{\partial v}{\partial r} + (1 - \mu^2) \frac{v}{r}}$$

$$Q(r, \mu) = \mu^2 \frac{\partial v}{\partial r} + (1 - \mu^2) \frac{v}{r}$$

symmetric in $\mu$

radial  

| tangential | velocity gradient |

$$Q(r, \mu) = \frac{v}{r} (1 + \sigma \mu^2)$$

$$\sigma = \frac{\partial \ln v}{\partial \ln r} - 1$$

departure from homologous expansion

$$v \propto r$$

$$\tau_s \approx R_* k \Delta l_s$$

$$\Delta l_s = \frac{1}{Q(r, \mu)}$$

geometrical width of interaction region (dimensionless)
5. The Sobolev Approximation

we made approximation

$$\tau(x, p, z) \approx R_* k(r(z_0, p)) \int \phi(x - \mu \nu) d\tilde{z}$$

only justified, if $k(r) \sim \text{const}$ over peak of integrand
width of peak is

$$\Delta l_s = \frac{1}{Q(r, \mu)}$$

$k(r)$ varies with density $\rho$ or on comparable scale

$$\Delta l_{dyn} = \left| \frac{d\rho}{dr} \frac{1}{\rho} \right|^{-1}$$

approximation is ok, if

$$\Delta l_s \ll \Delta l_{dyn}$$

Sobolev approximation
in winds (as we will learn later)

\[ \rho(r) \propto r^{-n}, \quad n = 1\ldots4 \]

\[ \Delta l_{\text{dyn}} = \frac{r}{n} \]

on the other hand

\[ \Delta l_s \approx \frac{r}{v} \]

if \( v \gg n \)

which is the case for strongly supersonic winds with \( v \gg 1 \)

Sobolev approximation is ok
6. Radiative transfer

solution of transfer eq. in (p,z) - geometry

\[
\frac{1}{R_\star} \frac{dI(x, p, z)}{dz} = \kappa_x(r(p, z)) \{I(x, p, z) - S(r(p, z))\}
\]

\[
\kappa_x = \kappa_x^{bg} + n_E \sigma_E + k(r) \phi(x - \mu v)
\]

background opacity/emissivity

Thomson scattering

line absorption/emission

\[
\epsilon_x = \epsilon_x^{bg} + n_E \sigma_E J_\nu + \epsilon(r) \phi(x - \mu v)
\]

\[
S(r) = \frac{\kappa_x^{bg}}{\kappa_x} S_{bg}(r) + \frac{n_E \sigma_E}{\kappa_x} J_\nu + \frac{k(r)}{\kappa_x} S_L(r)
\]

total source function

\[
S_{bg}(r) = \frac{\epsilon_x^{bg}}{\kappa_x^{bg}}, \quad S_L = \frac{\epsilon(r)}{k(r)}
\]

total absorption coefficient

total emission coefficient
formal solution along ray with $p = \text{const.}$

\[
I(x, p, z) = R_\star \int_{z_{\min}}^{z} S(r(p, \tilde{z})) e^{-\Delta \tau(x, p, \tilde{z})} \kappa_x(r(p, \tilde{z})) d\tilde{z} + I_{z_{\min}} e^{-\tau(x, p, z)}
\]

\[
\tau(x, p, z)
\]

\[
\Delta \tau(x, p, z)
\]

\[
\tau(x, p, z) = R_\star \int_{z_{\min}}^{z} \kappa_x(r(p, \tilde{z})) d\tilde{z} \quad \text{optical path} \ z_{\min} \text{ to } z
\]

\[
\Delta \tau(x, p, \tilde{z}) = \tau(x, p, z, ) - \tau(x, p, \tilde{z}) \quad \text{optical path} \ \tilde{z} \text{ to } z
\]
discussion of formal solution

\[ I(x, p, z) = R_* \int_{z_{\text{min}}}^{z} S(r(p, \tilde{z})) e^{-\Delta \tau(x, p, \tilde{z})} \kappa_x(r(p, \tilde{z})) d\tilde{z} + I_{z_{\text{min}}} e^{-\tau(x, p, z)} \]

- \( I_{z_{\text{min}}} \)
- \( I_{z_{\text{min}}} e^{-\tau(\tilde{z})} \)
- \( S(\tilde{z}) \kappa_x(\tilde{z}) = \epsilon_x(\tilde{z}) \)
- \( S(\tilde{z}) \kappa_x(\tilde{z}) e^{-\Delta \tau(\tilde{z})} \)

incident radiation at initial point
fraction of incident radiation arriving at \( z \) weakened by absorption
re-emitted intensity at \( \tilde{z} \)
fraction of intensity emitted at \( \tilde{z} \) arriving at \( z \); weakened by absorption between \( \tilde{z} \) and \( z \)

\[ R_* \int_{z_{\text{min}}}^{z} ... d\tilde{z} \]
Integration of contributions from all \( \tilde{z} \)
\[ I(x, p, z) = R_* \int_{z_{\text{min}}}^{z} S(r(p, \tilde{z})) e^{-\Delta \tau(x, p, \tilde{z})} \kappa_x(r(p, \tilde{z})) d\tilde{z} + I_{z_{\text{min}}} e^{-\tau(x, p, z)} \]

\[ I_{z_{\text{min}}}(p) \quad \text{depends on impact parameter } p \]

\[ 0 \leq |p| \leq 1 \quad I_{z_{\text{min}}}(p) = I_c(p) \quad \text{continuum intensity from stellar core} \]

\[ 1 \leq |p| \quad \text{integration starts at } z = -\infty \]

\[ I_{z_{\text{min}}}(p) = 0 \]
With $\kappa_x(r)$ and $S(r)$ known we can calculate intensity arriving at observer observer sees integral over all p weighted by area from ring with $\sim 2\pi dp$

$$I(x, p, z = z_{max}) \int \sum \text{flux observer sees}$$

continuum flux from stellar surface

$$F(x) \propto R_*^2 \int_0^{\infty} I(x, p, z_{max}) 2\pi dp$$

observed normalized line profile

$$P(x) = \frac{\int_0^{\infty} I(x, p, z = z_{max}) p dp}{\int_0^{1} I_c(p) dp}$$

$$F_{cont}(x) \propto R_*^2 \int_0^{1} I_c(x, p) 2\pi dp$$
standard stellar wind diagnostics

\[ \kappa_x(r) \]
\[ S(r) \]

from solution of NLTE rate equations

then numerical solution to calculate line profile \( P(x) \)

→ spectral diagnostics of line profiles

→ determination of \( v(r), \rho(r), \dot{M} \)
for better understanding of wind diagnostics

2 approximations in the following

1. Line opacity dominates

\[ k(r) \gg \kappa^b_{\nu}, n_E \sigma_E \]

2. assumption \( \frac{dv}{dr} \) large

\[ \rightarrow \text{ Sobolev approximation} \]

radiative transfer only in interaction zones
Approximation 1

\[ \kappa_x = k(r) \phi(x - \mu v) \quad \epsilon_x = \epsilon(r) \phi(x - \mu v) \]

\[ S(r) = S_L(r) = \frac{\epsilon(r)}{k(r)} \]

with exakt expression for \( \epsilon(r), k(r) \)

\[ k(r) = \frac{\pi e^2}{mc} f_{lu} \left\{ n_l - \frac{g_l}{g_u} n_u \right\} \frac{1}{\Delta \nu_D} \]

\[ \epsilon(r) = \frac{\pi e^2}{mc} f_{lu} \frac{2h \nu^3}{c^2} \frac{g_l}{g_u} n_u \frac{1}{\Delta \nu_D} \]

\[ \Rightarrow \quad S_L(r) = \frac{2h \nu^3}{c^2} \frac{1}{\frac{g_l}{n_l} / \frac{g_u}{n_u}} - 1 \]
formal solution in p-z geometry (p. 28)

\[ I(x, p, z) = R_* \int_{z_{\text{min}}}^{\tilde{z}} S(r(p, \tilde{z})) e^{-\Delta \tau(x, p, \tilde{z})} k_x(r(p, \tilde{z})) d\tilde{z} + I_{z_{\text{min}}} e^{-\tau(x, p, z)} \]

changes to

\[ I(x, p, z) = R_* \int_{z_{\text{min}}}^{\tilde{z}} S_L(r) e^{-\Delta \tau(x, p, \tilde{z})} k(r) \phi(x - \mu v) d\tilde{z} + I_{z_{\text{min}}} e^{-\tau(x, p, z)} \]

\[ \tau(x, p, z) = R_* \int_{z_{\text{min}}}^{\tilde{z}} k(r) \phi(x - \mu v) d\tilde{z} \]

\[ \Delta \tau(x, p, \tilde{z}) = \tau(x, p, z) - \tau(x, p, \tilde{z}) \]

\[ r = r(p, \tilde{z}) \]

\[ \mu = \mu(p, \tilde{z}) \]
Approximation 2 see pages 20 to 26

\[ \tau(x, p, z) \approx \tau_s(p, z_0) \Phi[u(x, p, z)] \]

\[ \Phi \approx 0, \quad z < z_0 \quad \Phi \approx 1, \quad z \geq z_0 \]

\[ R_\ast \int_{z_{\text{min}}}^{z} S_L(r) e^{-\Delta \tau(x, p, \tilde{z})} k(r) \phi(x - \mu v) d\tilde{z} \]

\[ \approx R_\ast S_L(r(p, z_0)) \int_{z_{\text{min}}}^{z} e^{-\Delta \tau(x, p, \tilde{z})} k(r) \phi(x - \mu v) d\tilde{z} \]

substituting \( \tilde{z} \to \Delta \tau(\tilde{z}) \)

\[ \Delta \tau(\tilde{z}) = \tau(z) - \tau(\tilde{z}) \]

\[ d(\Delta \tau) = -R_\ast k(\tilde{z}) \phi(x - \mu v) d\tilde{z} \]

\[ \Delta \tau(z) = 0 \quad \Delta \tau(z_{\text{min}}) = \tau(z) = \tau_s \Phi \]

\[ = S_L(r_0(p, z_0)) \int_{0}^{\tau_s \Phi} e^{-\Delta \tau} d\Delta \tau = S_L(r_0(p, z_0))(1 - e^{-\tau_s \Phi}) \]

note: \( z_0 \) is \( z \)-value of interaction zone along \( p = \text{const.} \)
for frequency \( x \)
Remember: general shape of interaction surfaces

now we can calculate line profile by integrating over pdp along interaction surface
Intensity determined by \( S_L(r_0), \tau_S(r_0) \) at interaction zone

\[
I(x, p, z) = S_L(r_0(p, z_0))(1 - e^{-\tau_S(p, z_0)}\Phi(z)) + I_{z_{min}} e^{-\tau_S(p, z_0)}\Phi(z)
\]

with,

\[
I_{z_{min}}(p) = \begin{cases} 
0 & 1 \leq |p| \\
I_c(p) & 0 \leq |p| \leq 1
\end{cases}
\]

Intensity arriving at telescope has \( \Phi(z \geq z_0) = 1 \)

\[
x \geq 0 \quad \text{blue frequencies}
\]

\[
I(x, p, z = \infty) = S_L(r_0(p, z_0))(1 - e^{-\tau_S(p, z_0)}) + I_c e^{-\tau_S(p, z_0)} \quad 0 \leq |p| \leq 1
\]

\[
I(x, p, z = \infty) = S_L(r_0(p, z_0))(1 - e^{-\tau_S(p, z_0)}) + 0 \quad |p| \geq 1
\]

\[
x \leq 0 \quad \text{red frequencies}
\]

\[
I(x, p, z = \infty) = I_c(p) \quad 0 \leq |p| \leq 1
\]

\[
I(x, p, z = \infty) = S_L(r_0(p, z_0))(1 - e^{-\tau_S(p, z_0)}) \quad |p| \geq 1
\]
Remember: general shape of interaction surfaces

now we can calculate line profile by integrating over pdp along interaction surface
\[ \beta = 1 \]

\[ \frac{x}{v_\infty} = -0.9, -0.8, -0.7 \ldots +0.7, +0.8, +0.9 \]
If we neglect limb-darkening of continuum radiation from stellar photosphere, then $I_c(p) = I_c = \text{const.}$

$$\int_0^1 I_c \ p \ dp = \frac{1}{2} I_c$$

$x \geq 0$  blue frequencies

$$P(x) = \frac{2}{I_c} \left\{ \int_0^\infty S_L(r_0)(1 - e^{-\tau_S(r_0)})pdp + I_c \int_0^1 e^{-\tau_S(r_0)}pdp \right\}$$

$$= \int_0^\infty \frac{S_L(r_0)}{I_c} (1 - e^{-\tau_S(r_0)})2pdp + \int_0^1 e^{-\tau_S(r_0)}2pdp$$

$x \leq 0$  red frequencies

$$P(x) = \frac{2}{I_c} \left\{ \int_0^1 I_c p dp + \int_1^\infty S_L(r_0)(1 - e^{-\tau_S(r_0)})p dp \right\}$$

$$= 1 + \int_1^\infty \frac{S_L(r_0)}{I_c} (1 - e^{-\tau_S(r_0)})2p dp$$
\[
P(x) = \int_0^\infty \frac{S_L(r_0)}{I_c} (1 - e^{-\tau_S(r_0)})2pd\rho + \int_0^1 e^{-\tau_S(r_0)}2pd\rho
\]

blue emission integral from 0 to \(\infty\)

\[
P(x) = 1 + \int_1^\infty \frac{S_L(r_0)}{I_c} (1 - e^{-\tau_S(r_0)})2pd\rho
\]

red emission integral from 1 to \(\infty\)
with definitions:

\[ P_{em}(x) = \int_{1}^{\infty} \frac{S_{L}}{I_{c}}(p, z_0)(1 - e^{-\tau_{S}(p, z_0)})2pd\!p \]

\[ P_{abs}(x) = \int_{0}^{1} \{ \frac{S_{L}}{I_{c}}(p, z_0)(1 - e^{-\tau_{S}(p, z_0)}) + e^{-\tau(p, z_0)} \}2pd\!p \]

→ line profiles

\[ x \leq 0 \quad \text{red} \quad x \geq 0 \quad \text{blue} \]

\[ P(x) = 1 + P_{em}(x) \quad P(x) = P_{abs}(x) + P_{em}(x) \]


**Discussion**

1. **Red part of line profile** \( x < 0 \)
   - Since \( P_{em}(x) > 0 \) \( \rightarrow \) \( P(x) > 1 \)
   - Always in emission!!!!!

2. **Blue part of line profile** \( x > 0 \)
   - \( S_L/I_c < 1 \) \( \rightarrow \) \( P_{abs}(x) < 1 \)
   - \( = 1 \)
   - \( > 1 \)

\( S_L/I_c \) determines symmetry of line profile
- Red/blue equal for \( S_L/I_c = 1 \) (good test of programs)
3. If $S_L/I_c < 1$ and if in addition $P_{\text{abs}}(x) + P_{\text{em}}(x) < 1$

$\rightarrow$ blue absorption!!!!

$P_{\text{abs}}(x) = (P_{\text{abs}}(x) + P_{\text{em}}(x)) - P_{\text{em}}(x)$

$\rightarrow$ disentangling of emission and absorption integral possible!!!
wind tomography possible!!!!
wind tomography
$\beta = 1$

$x/v_\infty = -0.9, -0.8, -0.7 \ldots +0.7, +0.8, +0.9$
Examples of stellar wind diagnostics

Example 1: Hα in O-stars

ionization from ground or excited level
cascade of subsequent spontaneous emissions
detailed NLTE calculations show $n_3/n_2 \sim \text{const. through winds}$

$S_L(r) = \frac{2h\nu^3}{c^2} \frac{1}{\frac{n_2}{n_3} \frac{g_3}{g_2} - 1} \approx \text{const.}$

$S_L/I_c \sim \text{const. We adopt } S_L/I_c \sim 1 \text{ for simplicity}$

$P_{\text{abs}}(x) = 1$

emission line with similar red and blue part
\[ P(x) = 1 + P_{em}(x) \] strength of emission determined by \( P_{em}(x) \)

\[
P_{em}(x) = \int_{1}^{\infty} \frac{S_L}{I_c} (p, z_0) (1 - e^{-\tau_S(p, z_0)}) dp
\]

\[ \approx \int_{1}^{\infty} \tau_S(p, z_0) dp \]

For O-stars \( \tau^H_S \ll 1 \) optically thin interaction region

\[
P_{em}(x) \approx \int_{1}^{\infty} \tau_S(p, z_0) dp
\]

\[
\tau_S = R_* \frac{k(r)}{\mu^2 \frac{dv}{dr} + (1 - \mu^2) \frac{v}{r}} \propto R_* k(r)
\]

\[
k(r) = \frac{\pi e^2}{mc} f_{lu} n_2(r) \left\{ 1 - \frac{n_3 g_2}{n_2 g_3} \right\} \frac{1}{\Delta \nu_D} \propto n_2(r)
\]

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detailed NLTE calculations show $n_2 \sim n_E n_p \sim \rho^2(r)$

$k(r) \sim \rho^2(r)$ with equation of continuity

$$M = R_*^2 4\pi r^2 \rho u$$

$$\rho^2(r) = \left(\frac{M}{R_*^2}\right)^2 \frac{1}{r^4 v^2}$$

$$P_{em}(x) \approx \frac{S_L}{I_c} \int_1^\infty \tau_S(p, z_0) 2p dp$$

$$\tau_s = R_* \frac{k(r)}{\mu^2 \frac{\partial v}{\partial r} + (1 - \mu^2) \frac{v}{r}}$$

$$P_{em}(x) \propto \left(\frac{M}{R_*^{3/2}}\right)^2 \int_1^\infty \frac{1}{r^4 v^2(r)} \frac{1}{\mu^2 \frac{\partial v}{\partial r} + (1 - \mu^2) \frac{v}{r}} 2p dp$$

Strength of stellar wind emission depends on $H_\alpha$ excellent mass-loss indicator

$$\frac{\dot{M}}{R_*^{3/2}}$$
$H\alpha$ as mass-loss indicator

dashed profile calculated with 25% higher mass-loss rate

exact integration of $P_{em}(x)$ along interaction surfaces for each frequency $x$ yields line profile

velocity field adopted was $v(r) = v_\infty \left(1 - \frac{1}{r}\right)^\beta$, $\beta = 2/3$
**H\textsubscript{\alpha} emission O-star**

Exact NLTE model calculation

Variation of \( \dot{M} \) by \( \pm 20\% \)

Kudritzki & Puls, 2000, AARA 38, 613

\[ R_{ji} = 4\pi \left( \frac{n_i}{n_j} \right) \int \frac{\sigma_{ij}}{h\nu} \left( \frac{2h\nu}{c^2} + J_\nu \right) \exp\left( -\frac{h\nu}{kT} \right) d\nu \]
$H_\alpha$ emission B supergiant - stellar wind


Diagram of HD 2905 with model calculation.
Example 2: strong UV resonance line \( \tau_S \gg 1 \)

resonance transition

absorption is followed by spontaneous emission

line scattering
detailed NLTE calculations show \( S_L(r) \approx \frac{1}{2} I_c \frac{1}{r^3} \)
is reasonable approximation

\[
P_{em}(x) = \int_1^\infty \frac{S_L}{I_c}(p, z_0)(1 - e^{-\tau_S(p, z_0)})2pdp
\]
\[
\approx \int_1^\infty \frac{1}{r^3}pdp
\]

in the same way

\[
P_{abs} \approx \int_0^1 \frac{1}{r^3}pdp
\]

P Cygni profile, because \( P_{abs}(x) + P_{em}(x) < 1 \)

but no information about mass-loss rates, since \( \tau_S \gg 1 \)

However the shape of velocity field \( v(r) \)
can be determined
exact integration of $P_{\text{em}}(x)$ and $P_{\text{abs}}(x)$ along interaction surfaces for each frequency $x$ yields line profile

$v(r) = v_\infty (1 - \frac{1}{r})^\beta$

solid $\beta = 1$

dashed $\beta = 3/2$
P Cygni profiles and $v_\infty$

Kudritzki & Puls, 2000, AARA 38, 613

Fit of $v_\infty$
± 5% accuracy