VII. Hydrodynamic theory of stellar winds

- observations: winds exist everywhere in the HRD

- hydrodynamic theory needed to describe stellar atmospheres with winds

Unified Model Atmospheres:
- based on the hydrodynamics of radiation driven winds
- fully self-consistent hydrodynamic transition from hydrostatic photosphere to supersonic winds
1. Hydrodynamical equations of an ideal compressible fluid

Definitions:

- \( \rho \vec{v} \): momentum density (momentum per unit volume)
- \( \vec{f} \): force per unit volume, \( \vec{f} dV \) acts on dV
- \( \Pi \): momentum flux tensor
- \( \Pi_{i,j} \): flux of momentum of component i perpendicular to direction j

\[
\Pi_{i,j} = \rho v_i v_j + p \delta_{ij}
\]

- Macroscopic momentum flow
- Stochastic motion
- p = gas pressure

eq. of mass continuity

\[
\frac{\partial}{\partial t} \int_V \rho dV = - \int_S \rho \vec{v} d\vec{S}
\]

\[
\int_S a d\vec{S} = \int_V \nabla a dV \quad \text{Gauss theorem}
\]

\[
\frac{\partial \rho}{\partial t} + \nabla (\rho \vec{v}) = 0
\]  
Eq. 1
momentum equation
\[ \frac{\partial}{\partial t} \int_V \rho \vec{v} dV = - \oint \Pi d\vec{S} + \int_V \vec{f} dV \]

change of volume
momentum with time
momentum flow through
borders of V
momentum change
through external forces

Gauss theorem
\[ \frac{\partial}{\partial t} (\rho \vec{v}) = \nabla \Pi + \vec{f} \]
definition \( \Pi \)
\[ \frac{\partial}{\partial t} (\rho \vec{v}) + \nabla (\rho \vec{v} \cdot \vec{v}) = - \nabla p + \vec{f} \]

**note**: for any scalar or vector
\[ \frac{\partial (\rho \alpha)}{\partial t} + \nabla (\rho \alpha \vec{v}) = \alpha \left\{ \frac{\partial \rho}{\partial t} + \nabla (\rho \vec{v}) \right\} + \rho \left\{ \frac{\partial \alpha}{\partial t} + (\vec{v} \cdot \nabla) \alpha \right\} = \rho \frac{D\alpha}{Dt} \]

\[ \frac{D\vec{v}}{Dt} = \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = - \nabla p + \vec{f} \]

Eq. 2

substantial derivative
in co-moving frame

Eq. **
\[ = 0 \quad \text{see, Eq. 1} \]
energy equation for internal energy per unit mass for an ideal gas

\[ e = \frac{p}{\rho \gamma - 1} = \frac{3}{2} \frac{k}{\mu m_p} T \]

1st law of thermodynamics for volume element \( dV \) with mass \( dm = \rho dV \)

\[ de + pd\left(\frac{1}{\rho}\right) = dq \]

work done per unit mass

external energy added or lost per unit mass

energy equation in co-moving frame

\[ \rho \left\{ \frac{De}{Dt} + p \frac{D(\frac{1}{\rho})}{Dt} \right\} = \rho \frac{Dq}{Dt} \]

Eq. 3a

\[ \dot{Q} = \rho \frac{Dq}{Dt} \]

Eq. 3b

external energy added or lost i.e., radiation, sound waves, convection, friction, magnetic fields, etc.
2. Stationary, spherically symmetric winds

\[ \rho(r, t) \rightarrow \rho(r) \quad \frac{\partial}{\partial t} = 0 \]

\[ \nabla \rightarrow \frac{d}{R_*dr} \]

\[ \vec{v} \cdot \nabla \rightarrow v \frac{d}{R_*dr} \]

\[ \nabla(\rho \vec{v}) \rightarrow \frac{1}{r^2} \frac{d}{R_*dr} r^2 \rho v \]

**Eq. 1** \[ \frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0 \]

**Eq. 4a, continuity** \[ \dot{M} = 4\pi R_*^2 r^2 \rho v \]

**Eq. 2** \[ \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right) = -\nabla p + \vec{f} \]

\[ \rho v \frac{dv}{R_*dr} + \frac{dp}{R_*dr} = f \]

**gravity** \[ f = -\rho \{g(r) - g_{\text{add}}(r)\} \]

**radiation, magnetic fields**

**Eq. 4b, momentum**
\[
\rho \left\{ \frac{De}{Dt} + p \frac{D(\frac{1}{\rho})}{Dt} \right\} = \dot{Q}
\]

Eq. 3

\[
\frac{D}{Dt} = \left\{ \frac{\partial}{\partial t} + (\vec{v} \cdot \nabla) \right\}
\]

\[
\rho v \frac{1}{R_*} \left\{ \frac{d}{dr} \frac{e}{\rho} + p \frac{d}{dr} \rho^{-1} \right\} = \dot{Q}
\]

\[
p \frac{d}{dr} \rho^{-1} = \frac{d}{dr} \left( \frac{p}{\rho} \right) - \frac{1}{\rho} \frac{d}{dr} p
\]

\[
e = \frac{p}{\rho} \frac{1}{\gamma - 1}
\]

\[
\rho v \frac{1}{R_*} \left\{ \frac{d}{dr} \frac{p}{\rho} \frac{\gamma}{\gamma - 1} + \rho^{-1} \frac{dp}{dr} \right\} = \dot{Q}
\]

Eq. 2

\[
\frac{v}{R_*} \frac{dp}{dr} = -v^2 \frac{dv}{R_* dr} - v \{g(r) - g_{add}(r)\}
\]

\[
v \frac{dv}{dr} = \frac{1}{2} \frac{dv^2}{dr}
\]

\[
\rho v \frac{1}{R_*} \frac{d}{dr} \left\{ \frac{1}{2} v^2 + \frac{p}{\rho} \frac{\gamma}{\gamma - 1} - \frac{GM_*}{R_*} \frac{1}{r} \right\} = \rho v g_{add}(r) + \dot{Q}
\]

Eq. 4c, energy
Eq. 4c $\rightarrow$ condition for existence of stellar winds

$$
\frac{d}{dr} \left\{ \frac{1}{2} v^2 + \frac{p}{\rho \gamma - 1} - \frac{GM_*}{R_*} \frac{1}{r} \right\} = R_* g_{add}(r) + \frac{R_*}{\rho v} \dot{Q}
$$

integration

$$
E(r) = \frac{1}{2} v^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} - \frac{GM_*}{R_*} \frac{1}{r} = E(r_0) + R_* \int_{r_0}^{r} g_{add}(r) \, dr + R_* \int_{r_0}^{r} \frac{\dot{Q}}{\rho v} \, dr
$$

select $r_0 \approx 1$ $\quad$ $E(r_0) = \frac{1}{2} v_0^2 + \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} - \frac{GM_*}{R_*} \approx -\frac{GM_*}{R_*}$

$$
\frac{GM_*}{R_*} = \frac{v_{esc}^2}{2}, \quad v_{esc} \approx 100...1000 km/s
$$

$$
\frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} = \frac{5}{2} v_{sound}^2, \quad v_{sound} \approx 5...20 km/s
$$

$v_0 \leq v_{sound}$
\[ \frac{1}{2}v^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} - \frac{GM_*}{R_*} \frac{1}{r} \approx -\frac{GM_*}{R_*} + R_* \int_{r_0}^{r} g_{add}(r) \, dr + R_* \int_{r_0}^{r} \frac{\dot{Q}}{\rho v} \, dr \]

\[ r \rightarrow \infty \]

\[ \frac{1}{2}v^\infty + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \approx -\frac{GM_*}{R_*} + R_* \int_{r_0}^{\infty} g_{add}(r) \, dr + R_* \int_{r_0}^{\infty} \frac{\dot{Q}}{\rho v} \, dr \]

left side only > 0, if at least one of the integrals \( \gg 0 \)

stellar wind can only escape potential well of the star, if

- significant momentum input

or

- significant energy input
3. “Naïve” theory without additional forces

first simple approach: isothermal wind without additional forces driven by gas pressure only

\[ g_{add}(r) = 0 \]

- example: solar wind

significant \( \dot{Q} \) in corona

\( T_{\text{corona}} \approx 10^6 K \)

pressure driven wind

Eq. 4b

\[ v \frac{dv}{dr} = -\frac{1}{\rho} \frac{dP}{dr} - \frac{GM_*}{R_*} \frac{1}{r^2} \]

with

\[ P = v_s^2 \rho \]

\[ \mu = \frac{1}{2} \]

pure ionized H

\[ \mu = 0.6...0.65 \]

hot solar gas

\[ v_s^2 = \frac{kT}{\mu m_p} \]

\[ v_{esc}^2 = 2 \frac{GM_*}{R_*} \]
\[ v \frac{dv}{dr} = -\frac{v_s^2}{\rho} \frac{d\rho}{dr} - \frac{dv_s^2}{dr} - \frac{1}{2} v_{esc} \frac{1}{r^2} \]

starting point to derive equation of motion for \( v(r) \)

next step: replace \( \rho \)

using Eq. 4a \[ \dot{M} = 4\pi R_*^2 r^2 \rho v \]

\[ \rho(r) = \frac{\dot{M}}{4\pi R_*^2} \frac{1}{r^2 v} \]

\[ \frac{d\rho}{dr} = \frac{\dot{M}}{4\pi R_*^2} \{ -\frac{2}{r^3 v} - \frac{1}{r^2 v^2} \frac{dv}{dr} \} = -\rho \{ \frac{2}{r} + \frac{1}{v} \frac{dv}{dr} \} \]

Eq. 5, density/velocity relationship

Eq. 5 can be used to replace density in eq. *
Eq. 6, equation of motion

\[ \{1 - \frac{v_s^2}{v^2}\}v \frac{dv}{dr} = \frac{1}{r} \left\{2v_s^2 - \frac{1}{2}v_{esc}^2 \frac{1}{r}\right\} - \frac{dv_s^2}{dr} \]

differential eq. for v(r) of pressure driven winds has singularity called the critical point at

\[ v^2(r_s) = v_s^2 \]  
Eq. 7a

\[ r_s = \frac{v_{esc}^2}{4v_s^2} \]  
Eq. 7b

left side Eq. 6 zero at critical point \( \rightarrow \) right side must also be zero \( T \sim \text{const} \rightarrow v_s \sim \text{const} \).

because \( v = v_s \) at \( r = r_s \), the critical point is also called the sonic point
Example: solar wind of solar corona
photospheric escape velocity of sun

\[
\nu_{\text{esc}} = 617 \text{km/s} \quad T_{\text{corona}} = 2 \times 10^6 \text{K} \implies v_s = 165 \text{km/s}
\]

\[\implies r_s = 3.5 \quad \text{sonic point in outer corona}\]

Note: if temperature of corona were as cool as photosphere,

\[
T_{\text{corona}} \approx T_{\text{photosphere}} \implies v_s = 8.5 \text{km/s}
\]

\[\implies r_s = 1320 \quad \text{we would still have a solar wind, but supersonic just at 6 AU}\]

in principle, all stars will have a (weak) wind, but the solar wind is strong because of high temperature in corona
the importance of the critical point
differential eq. 6 has several families of solutions
critical point \(\rightarrow\) disentangle topology of solutions
\(\rightarrow\) find physically relevant solution

for simplicity, \(v_s = \text{const.}\) with \(\frac{dv}{dr} = \frac{v_s^2}{2} \frac{d(v_s^2)}{dr}\)

re-write Eq. 6

\[
\left\{1 - \frac{v_s^2}{v^2}\right\} \frac{d(v_s^2)}{dr} = \frac{4}{r} \left\{1 - \frac{r_s}{r}\right\}
\]

\[
\frac{v^2}{v_s^2} - ln\left(\frac{v^2}{v_s^2}\right) = 4ln\left(\frac{r}{r_s}\right) + 4\frac{r_s}{r} + C
\]

the implicit solution Eq. 8b for \(v(r)\) together with
the conditions for the critical point Eq. 7a,b can be used to discuss topology of solution families
\[ \frac{v^2}{v_s^2} - ln \frac{v^2}{v_s^2} = 4ln \frac{r}{r_s} + 4 \frac{r_s}{r} + C \]

\[ \{1 - \frac{v_s^2}{v^2}\} \frac{dv^2}{dr} = \frac{4}{r} \{1 - \frac{r_s}{r}\} \]

**Case I**  \( v(r) \) exists for \( 1 \leq r \leq \infty \) and is transsonic, i.e.

\( r = r_s : \ v(r_s) = v_s \) and \( \frac{dv^2}{dr} \neq 0 \)

All monotonic solutions \( 8b \) with \( C = -3 \)

**What is the value of** \( \frac{dv^2}{v_s^2} \) **at sonic point?**

**Eq. 8a**

\[ \{1 - \frac{v_s^2}{v^2}\} \frac{dv^2}{v_s^2} = \frac{4}{r} \{1 - \frac{r_s}{r}\} \]

\[ \frac{dv^2}{v_s^2} \quad dr = \frac{4}{r} \frac{1 - r_s}{1 - \frac{v_s^2}{v^2}} \]

L’Hospital:

\[ \frac{d}{dr} \frac{4}{r} \{1 - \frac{r_s}{r}\} = -\frac{4}{r^2}(1 - \frac{r_s}{r}) + \frac{4r_s}{r^3} \rightarrow r_s \frac{4}{r_s^2} \]

\[ \frac{d}{dr} \{1 - \frac{v_s^2}{v^2}\} = \frac{v_s^2}{v^4} \frac{dv^2}{v_s^2} \quad dr \rightarrow r_s \]

\[ \left[ \frac{dv^2}{dr} \right]^2 = \frac{4}{r_s^2} \]

**Eq. 9a**

**two solutions possible**
case I, 1
\[
\frac{d\frac{v^2}{v_s^2}}{dr} = \frac{2}{r_s} = \frac{8v_s^2}{v_{esc}^2}
\]
transonic, monotonic, increasing
this solution is realized in solar wind

case I, 2
\[
\frac{d\frac{v^2}{v_s^2}}{dr} = -\frac{2}{r_s} = -\frac{8v_s^2}{v_{esc}^2}
\]
transonic, monotonic, decreasing
this solution is not realized

sketch of
2 solutions

\[\frac{v^2}{v_s^2} - \ln\frac{v^2}{v_s^2} = 4\ln\frac{r}{r_s} + 4\frac{r_s}{r} - 3\]

I, 1: \(v^2 \geq v_s^2\) for \(r \geq r_s\)
\[
v \approx 2\ln\frac{r}{r_s}^{1/2}
\]
\[
v \approx e^{-2\frac{r_s}{r}} = e^{-\frac{1}{\frac{r}{r_s}}}
\]

I, 2: \(v^2 \leq v_s^2\)
\(r \leq r_s\)
\[
-v \approx 4\ln\frac{r}{r_s}^{1/2} \approx 2v_s r^{-1/2}
\]

\[v \approx \frac{1}{r^2}\]
also possible: case II  

\[ v(r) \text{ exists for } 1 \leq r \leq \infty, \text{ but } \]

\[ r = r_s : v(r_s) = v_s \text{ and } \frac{dv^2}{dr} = 0 \]

2 possibilities

case II, 1:  \( v \geq v_s \text{ for } 1 \leq r \leq \infty, \ C > -3 \) supersonic solution

case II, 2:  \( v \leq v_s \text{ for } 1 \leq r \leq \infty, \ C < -3 \) subsonic solution

both cases not realized

however, II,2 was for a long time discussed as discussed as a possible option, until space observations showed that solar wind is supersonic
also not realized is case III non-unique solutions

\begin{align*}
in \quad & 1 \leq r \leq r_s \\
with \quad & r = r_s : v(r_s) \neq v_s \text{ and } \frac{dv^2}{dr} = \infty \\
or \quad & r_s \leq r \leq \infty
\end{align*}

note: discussion of solution topologies crucial to identify physically realized solution. We will repeat this, when dealing with the much more complicated case of radiation driven winds.
Discussion of the realistic solution:
trans-sonic, monotonic, increasing (Parker, 1958)

\[
\frac{v^2}{v_s^2} - \ln \frac{v^2}{v_s^2} = 4\ln \frac{r}{r_s} + 4\frac{r_s}{r} - 3
\]

\[
\frac{v^2}{v_s^2} e^{-\frac{v^2}{v_s^2}} = \left(\frac{r_s}{r}\right)^4 e^{-4\frac{r_s}{r}} + 3
\]

Limiting cases:

1. \(r \ll r_s, v \ll v_s\):
   \[
   v = v_s \left(\frac{r_s}{r}\right)^2 e^{-2r_s r} + \frac{3}{2}
   \]

2. \(r \gg r_s, v \gg v_s\):
   \[
   v = 2v_s \left\{\ln \frac{r}{r_s}\right\}^{1/2}
   \]

\(v \to \infty\) if \(r \to \infty\), consequence of the unrealistic assumption \(T = \text{const.}\), i.e. pressure can accelerate until infinity.
In reality, \(T\) drops and \(v\) remains finite.
density stratification

Eq. 4a \( \dot{M} = 4\pi R_*^2 r^2 \rho v \)

\( \rho(r) = \rho_1 v_1 \frac{1}{r^2 v} \)

Eq. 12

\[
\frac{\rho^2}{\rho_1^2} \frac{v_s^2}{v_1^2} e^{-\frac{v_s^2}{v_1^2}} \frac{1}{r^4} \frac{\rho^2}{\rho_1^2} \frac{v_s^2}{v_1^2} = \frac{1}{r_s^4} e^{4\frac{r_s}{r} - 3}
\]

solution \( 1 < r < \infty \)

for \( r \ll r_s \) \( \frac{\rho}{\rho_1} \frac{v_s}{v_1} = \frac{1}{r_s^2} e^{2\frac{r_s}{r} - \frac{3}{2}} = \frac{1}{r_s^2} e^{\frac{v_{esc}^2}{2v_s^2} \frac{1}{r} - \frac{3}{2}} \)

hydrostatic solution (Homework 1)

for \( r \gg r_s \) \( \frac{\rho}{\rho_1} \frac{v_s}{v_1} = \frac{1}{2r_s^2} (\ln \frac{r}{r_s})^{-\frac{1}{2}} \)

\( \rho < \rho_{static} \)
mass-loss rate

if $\rho_1$ at bottom of wind known $\Rightarrow$ with $v_1$ (eq. 11)

$$\dot{M} = 4\pi R_*^2 \rho_1 v_1$$

for $\rho_1 = 10^{-14} \text{ g/cm}^3$

<table>
<thead>
<tr>
<th>$T_{\text{corona}}$</th>
<th>$r_s$</th>
<th>$M_{\odot}/\text{yr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \times 10^6 \text{K}$</td>
<td>6.9</td>
<td>$1.6 \times 10^{-14}$</td>
</tr>
<tr>
<td>2</td>
<td>3.5</td>
<td>$3.7 \times 10^{-12}$</td>
</tr>
<tr>
<td>3</td>
<td>2.3</td>
<td>$8.2 \times 10^{-11}$</td>
</tr>
</tbody>
</table>

observed value is $2 \times 10^{-13} \ M_{\odot}/\text{yr}$
THE VELOCITY AND DENSITY DISTRIBUTION

\[ \log \frac{v}{v_0} \]

\[ \log \frac{\rho}{\rho_0} \]

\[ \log r/r_0 \]

velocity

density

\[ T_0 = \frac{r_s}{10} \]
4. Additional forces: radiation

Pressure driven winds can explain solar wind (note: that in reality you need magnetic fields to explain coronal heating) but...

can winds of hot stars be driven by pressure only?

crucial observational facts:

- cool winds, no corona: $T_{\text{wind}} \sim T_{\text{eff}} \sim 40000K$  
  $v_s \sim 20$ km/s
- sonic point $v = v_s$ very close to photosphere at $r \sim 1.05$
- $v_{\text{esc}} \sim 500 \ldots 1000$km/s

with pressure driven wind theory

$$r_s^{\text{theory}} = \frac{v_{\text{esc}}^2}{4v_s^2} \approx 100\ldots2500$$

we need additional force $g_{\text{add}}$ !!!
radiative acceleration

hot stars $B_{\nu}(T)$ large because $T$ large
$I_{\nu}$, $J_{\nu}$ large
large amount of photon momentum
photon absorption transfers momentum to atmospheric plasma

calculation of $g_{\text{add}} = g_{\text{extra}}$:

\[
grad = \frac{\text{absorbed momentum}}{\text{time} \cdot \text{mass}}
\]

absorbed energy per unit time:

\[
\kappa_{\nu} I_{\nu} d\nu d\omega
\]

momentum per time & mass:

\[
\kappa_{\nu} \frac{I_{\nu}}{c} d\nu d\omega
\]

only projection into radial direction contributes

\[
grad(\mu, \nu, r) d\omega d\nu = \frac{\kappa_{\nu} I_{\nu}}{c \rho} \mu d\nu d\omega
\]
with azimuthal symmetry of radiation field and \( d\omega = 2\pi d\mu \)

\[
g_{rad}(r) = \frac{2\pi}{c\rho(r)} \int_{0}^{\infty} \int_{-1}^{1} \kappa_{\nu} I_{\nu}(r, \mu) d\mu d\nu
\]

Eq. 13

Eq. 13 exact, as long as all absorption processes included

\[
\kappa_{\nu} = n_{E} \sigma_{E} + \kappa_{\nu}^{c} + \kappa_{\nu}^{L}
\]

Thompson scattering
continuous absorption
line absorption

\[
g_{rad}(r) = g_{rad}^{Th} + g_{rad}^{c} + g_{rad}^{L}
\]

\[
g_{rad}^{Th} = \frac{4\pi}{c} \frac{n_{e} \sigma_{E}}{\rho} \int_{0}^{\infty} H_{\nu} d\nu = \frac{1}{c} \frac{n_{E} \sigma_{E}}{\rho} \frac{L}{4\pi r^{2} R_{*}^{2}}
\]

Eq. 14

\[
g_{rad}^{c} = \frac{4\pi}{c\rho} \int_{0}^{\infty} \kappa_{\nu} H_{\nu} d\nu
\]

\[
g_{rad}^{L} = \frac{4\pi}{c\rho} \int_{0}^{\infty} \int_{-1}^{1} \kappa_{\nu}^{L}(\mu, v(r)) \mu I_{\mu}(\mu) \nu d\mu d\nu
\]

Eq. 15

remember:

\[
H_{\nu} = \frac{1}{2} \int_{-1}^{1} \mu I_{\nu}(\mu) d\mu \quad H = \int_{0}^{\infty} H_{\nu} d\nu \quad L = 4\pi R_{*}^{2} r^{2} (4\pi H)
\]
corollary: approximation for $\tau_\nu \gg 1$ at all frequencies

radiation field thermalized at all frequencies

“diffusion approximation”

$$H_\nu = \frac{1}{2} \int_{-1}^{1} \mu I_\nu(\mu) d\mu = \frac{1}{3} \frac{1}{\kappa_\nu} \frac{dB_\nu(T)}{dr}$$

$$g_{rad}(r) = \frac{4\pi}{c\rho} \frac{d}{dr} \frac{1}{3} \int_{0}^{\infty} B_\nu(T) d\nu = \frac{4\pi}{c\rho} \frac{d}{dr} \frac{1}{3} \sigma_B \frac{dT^4}{dr}$$

$$g_{rad}(r) = \frac{1}{\rho} \frac{d}{dr} P_{rad}$$

$$P_{rad}(r) = \frac{1}{3} \frac{4\pi}{c} \sigma_B T^4(r)$$

“radiation pressure”
in hot star winds the influence of $g_{rad}$ is usually small but

**influence of Thomson scattering**

$$ g_{rad}^c = \frac{1}{c} \frac{n_E \sigma_E}{\rho} \frac{L}{4\pi r^2 R_*^2} $$

with $n_E = n_p (1 + qY)$, $Y = \frac{n_{He}}{n_H}$, $\rho = n_p m_p (1 + 4Y)$, $\frac{n_E}{\rho} = \frac{1}{m_p} \frac{1 + qY}{1 + 4Y}$

$q$ electrons provided by helium nucleus (0 ...2)

$$ g_{rad}^T = \frac{\sigma_E}{cm_p} \frac{1 + qY}{1 + 4Y} \frac{L}{4\pi r^2 R_*^2} $$

and with

$$ g(r) = G \frac{M_*}{R_*^2} \frac{1}{r^2} $$

$$ \Gamma_{Th} = g_{rad}^T / g(r) = 3.062 \cdot 10^{-5} \frac{1 + qY}{1 + 4Y} \frac{L}{L_\odot} \frac{M_\odot}{M_*} $$

Eq. 16

with Eq. 4b new momentum equation

$$ \nu \frac{dv}{R_* dr} = - \frac{1}{\rho} \frac{dP_{gas}}{R_* dr} - \frac{GM_*}{R_*^2} \frac{1}{r^2} (1 - \Gamma_{Th}) + g_{rad}^L $$

Eq. 17

Thomson scattering reduces gravity by constant factor
with \( \tilde{v}_{esc}^2 = v_{esc}^2(1 - \Gamma_{Th}) \) \hspace{1cm} \text{Eq. 18} \hspace{1cm} \text{effective escape velocity}

and eq. 6 we obtain new equation of motion \hspace{1cm} \text{Eq. 19}

\[
\left(1 - \frac{v_s^2}{v^2}\right)v \frac{dv}{dr} = \frac{1}{r} \left\{ 2v_s^2 - \frac{1}{2} \tilde{v}_{esc}^2 \frac{1}{r} \right\} - \frac{dv_s^2}{dr} + R_s g_r^L
\]

does Thomson scattering alone do the job?

if we ignore \( g_r^L, \frac{dv_s}{dr} \) \hspace{2cm} \text{sonic point at}

\[
r_s = \frac{\tilde{v}_{esc}^2}{4v_s^2} = \frac{v_{esc}^4}{4v_s^4} (1 - \Gamma_{Th}) \hspace{1cm} \text{Eq. 19b}
\]

typical \( \Gamma \)-value for hot supergiants \( \Gamma = 0.5 \)

sonic point for hot stars still at \( r_s \sim 50...100 \), but observed is \( r_s \sim 1.1 \)

need additional accelerating force!!!!
5. Very simplified description of line driven winds

As first step: very simplified description of \( g^{L}_{rad} \) assumptions:

- only radially streaming photons
- only lines with \( \tau_{s} \gg 1 \) contribute
- Sobolev approximation \( \Delta v \gg v_{th} \quad \Delta v = \frac{\Delta v}{c} \nu_{i} \gg \Delta \nu_{D} \)
- number of strong lines independent of \( r \)
shell in stellar wind: calculation of line $i$ at $\nu_i \rightarrow g_{rad}^i$

\[
g_{rad}^i = \frac{\text{abs. momentum}}{\Delta t \cdot \Delta m} \Delta m = 4\pi r^2 \rho \Delta r R_*^3
\]

\[
\frac{\text{abs. momentum}}{\Delta t} = \frac{L}{c} \frac{L_{\nu_i} \Delta \nu_i}{L}
\]

fraction abs. by line $i$ $\Delta \nu_i$, line width
$L_{\nu_i}$, luminosity at $\nu_i$

total photon momentum

$\Delta \nu_i = \frac{\Delta \nu}{c} \nu_i$, $\Delta \nu_i \gg \Delta \nu_D$

\[
g_{rad}^i = \frac{L}{c^2} \frac{L_{\nu_i} \nu_i}{L} \frac{1}{4\pi r^2 \rho R_*^3} \frac{1}{dr}
\]

contribution by all lines: $g_{rad}^L = \sum_i g_{rad}^i$

\[
g_{rad}^L = \frac{L}{c^2} \sum_i \frac{L_{\nu_i} \nu_i}{L} \frac{1}{4\pi r^2 \rho R_*^3} \frac{1}{dr}
\]

Eq. 20

line acceleration $\sim$ velocity gradient !!!

$N_{eff}$ effective number of strong lines
new equation of motion (T = const.)

\[
\left\{1 - \frac{v_s^2}{v^2}\right\}v \frac{dv}{dr} = \frac{1}{r} \left\{2v_s^2 - \frac{1}{2} \tilde{v}_{esc} \frac{1}{r}\right\} + \frac{L}{c^2 N_{eff}} \frac{1}{4\pi R_* r^2 \rho} \frac{dv}{dr}
\]

Eq. 21a

with

\[\dot{M} = 4\pi R_* r^2 \rho v\]

\[
\left\{1 - \frac{v_s^2}{v^2} - \frac{L}{c^2} \frac{N_{eff}}{\dot{M}} \right\} \frac{d}{dr} \left(\frac{v_s^2}{v^2}\right) = \frac{4}{r} \left\{1 - \frac{1}{4} \tilde{v}_{esc} \frac{1}{r}\right\}
\]

Eq. 21b

because \[g_{rad}^L \propto \frac{dv^2}{dr}\]  
no singularity at \[v = v_s\]

r.h.s. of Eq. 21b negative smaller than old \(r_s\)

\[r_s < \frac{1}{4} \frac{\tilde{v}_{esc}^2}{v_s^2}\]

we assume \[r_s \ll \frac{1}{4} \frac{\tilde{v}_{esc}^2}{v_s^2}\]

and prove later that this is correct
multiplying eq. 21a by $4\pi R_*^2 r^2 \rho$ and assuming $v^2 \gg v_s^2$

$$\dot{M} \frac{dv}{dr} = \frac{L}{c^2} N_{eff} \frac{dv}{dr} - \frac{1}{2} \tilde{v}_{esc}^2 4\pi R_*^2 \rho$$

$$\dot{M} = \frac{L}{c^2} N_{eff} \left(1 - \frac{\frac{1}{2} \tilde{v}_{esc}^2 4\pi R_*^2 \rho c^2}{\frac{dv}{dr} LN_{eff}} \right)$$

$$\dot{M}, L, N_{eff} = \text{const.} \implies \epsilon = \text{const.} \quad \frac{dv}{dr} \geq 0 \implies \epsilon \geq 0$$

$$\dot{M} = \frac{L}{c^2} N_{eff} \left\{1 - \epsilon \right\}$$

Eq. 21c

$0 \leq \epsilon \leq 1$

combining eq. 21b and c for $r \gg r_s$

$$\left\{1 - \frac{1}{1-\epsilon} \right\} \frac{dv^2}{dr} = -\tilde{v}_{esc}^2 \frac{1}{r^2}$$
\[ \frac{dv^2}{dr} = \frac{1 - \epsilon}{\epsilon} \tilde{v}_{esc} \frac{1}{r^2} \]

integrating from \( r_s \) to \( r \)

\[ v^2(r) = \frac{1 - \epsilon}{\epsilon} \tilde{v}_{esc} \left( \frac{1}{r_s} - \frac{1}{r} \right) + v_s^2 \]

\[ v_\infty = \frac{1 - \epsilon}{\epsilon} \tilde{v}_{esc} \]

from observations (Abbott, 1978, 1982)

\[ v_\infty = 3v_{esc} \implies \epsilon \approx 0.1 \]

\[ \dot{M} = \frac{L}{c^2} N_{eff}, \quad N_{eff}^{theory} \approx 50 \]

\[ \dot{M} \left[ M_\odot/yr \right] = 3.5 \cdot 10^{-12} \frac{L}{L_\odot} \]

\[ \text{Eq. 22 } \beta\text{-velocity field, } \beta = 0.5 \]
The graph shows the relationship between the terminal velocity ($V_\infty$) and the escape velocity ($V_{esc}$) in units of km sec$^{-1}$. The equation $V_{esc} = 3 \times \sqrt[4]{\frac{G M_e}{R_e}} \left(1 - \frac{1}{1 + \frac{1}{10}}\right)$ is provided for the escape velocity, where $G$ is the gravitational constant, $M_e$ is the mass of the object, and $R_e$ is its radius.
Simplified theory of line driven winds explains

- \( v_\infty \propto v_{esc} \) but does not give proportionality constant

- right order of magnitude for \( \dot{M} \)

  but predicts \( \dot{M} \propto L \)

whereas is observed \( \dot{M} \propto L^x, \ x = 1.7 \)

improved theory needed!!!