3. Stellar radial pulsation and stability

\[ m_V \]

\[ T_{\text{eff}} \]

spectral type

\[ v_{\text{rad}} \]

\[ \Delta R = \int v dt \]

\[ \delta \text{ Cephei} \]

stellar disk from \( \Delta R \)
Interferometric observations of δ Cep (Mérand et al. 2005)
instability strip

Hayashi limit of low mass fully convective stars

main sequence
the instability strip

stars in instability strip have significant layers not too deep, not too superficial, where stellar opacity increases with $T$.

This is caused by the excitation and ionization of H and He.

Consider a perturbation causing a local compression

- $T$ increase
- $\kappa$ increase
- perturbation heat not radiated away but stored
- expansion beyond equilibrium point
- then cooling and $T$ decreasing
- gravity pulls gas masses back
- cyclically pulsation

$$\kappa \sim T^\alpha, \alpha \geq 0$$
stars to the left of instability strip are hotter and H and He are ionized in the interior.

The opacity decreases with increasing $T$.

For a perturbation causing a local compression we have again

$\rightarrow$ $T$ increase

but now

$\rightarrow$ $\kappa$ decreases

$\rightarrow$ perturbation heat is quickly radiated away

$\rightarrow$ perturbation is damped, no pulsation

To the right of the instability strip the energy transport is dominated by very effective convection and not by radiation transport. The compression heat is, thus, effectively transported away and the opacity behavior is irrelevant.

$$\kappa \sim T^{-\alpha}, \alpha \geq 0$$
Fig. 41.2  An opacity surface ("κ mountain") for the outer layers of a star as in Fig. 17.6. But this time the dependence with respect to $P$ (in dyn cm$^{-2}$) and $T$ (in K) is shown. The dotted line corresponds to the stratification inside a Cepheid of 7$M_\odot$. The white areas of the "mountain" indicate regions which excite the pulsation and the black ones those which damp it. The excitation in the region of $\lg T \approx 4.6$ is due to the second ionization of helium.
a simple model of radial pulsation

in hydrostatic equilibrium

\[ \frac{dP_0}{dr_0} = G \frac{Mr_0}{r_0^2} \rho_0 \]

mass confined within \( r_0 \)

mass within \( dr_0 \)

\( r_0 \) radial coordinate in equilibrium

\( P_0 \) pressure

\( \rho_0 \) mass density

“Gedanken Experiment”: compress star and then release \( \rightarrow \) oscillation

\[ \kappa \sim T^\alpha, \alpha \geq 0 \] \( \rightarrow \) compression energy stored

\( \rightarrow \) no oscillation damping
pulsation leads to radial shift of mass shells within the stars:  \( \frac{\Delta r}{r_0} = x(t) \)

new - time dependent - coordinate of mass shells:  \( r(t) = r_0[1 + x(t)] \)

\( x(t) \) is a time dependent perturbation and we assume that the moving shells conserve their mass:

\[ \rho r^2 \partial r = \rho_0 r_0^2 \partial r_0 \]

with \( \partial r = [1 + x(t)]\partial r_0 \) we then obtain for the density:

\[ \rho = \rho_0[1 + x(t)]^{-3} \]

if the mass shells do not exchange energy, we have an adiabatic situation:

\[ P = P_0 \left( \frac{\rho}{\rho_0} \right)^\gamma \]

\[ P = P_0[1 + x(t)]^{-3\gamma} \]

because of the perturbation the original hydrostatic equation does not hold anymore. In the coordinate frame co-moving with \( \partial M_r \) we need to add the inertia term \( -\rho \frac{\partial^2 r}{\partial t^2} \)

\[ \frac{\partial P}{\partial r} = -G \frac{M_r}{r^2} \rho - \rho \frac{\partial^2 r}{\partial t^2} \] eom
small perturbation: $|x(t)| \ll 1$ and $M_r = M_{r_0}$

$$
\rho = \rho_0 [1 - 3x(t)] \\
P = P_0 [1 - 3\gamma x(t)] \\
\frac{1}{r^2} = \frac{1}{r_0^2} [1 - 2x(t)]
$$

left hand side of eom:

$$
\frac{\partial P}{\partial r} = \frac{\partial P}{\partial r_0} \frac{\partial r_0}{\partial r} = \frac{\partial P_0}{\partial r_0} \frac{1 - 3\gamma x}{1 + x} \approx \frac{\partial P_0}{\partial r_0} (1 - 3\gamma x)(1 - x) \\
= G \frac{M_{r_0}}{r_0^2} \rho_0 (1 - 3\gamma x)(1 - x) \\
\approx G \frac{M_{r_0}}{r_0^2} \rho_0 (1 - x(3\gamma + 1))
$$

neglecting $x^2$ terms
on right hand side of eom we have the terms

\[
G \frac{M_r}{r^2} \rho = G \frac{M_{r_0}}{r_0^2} (1 - 2x) \rho_0 (1 - 3x) \approx G \frac{M_{r_0}}{r_0^2} \rho_0 (1 - 5x)
\]

and

\[
-\rho \frac{\partial^2 r}{\partial t^2} = -\rho_0 (1 - 3x) r_0 \frac{\partial^2 x}{\partial t^2}
\]

left hand eom side equal to right hand side

\[
G \frac{M_{r_0}}{r_0^2} \rho_0 (1 - x (3\gamma - 1)) = G \frac{M_{r_0}}{r_0^2} \rho_0 (1 - 5x) + \rho_0 (1 - 3x) r_0 \frac{\partial^2 x}{\partial t^2}
\]

\[
r_0 \frac{\partial^2 x}{\partial t^2} = G \frac{M_{r_0}}{r_0^2} \frac{4 - 3\gamma}{1 - 3x} x(t)
\]
At the outer edge of the star $r_0 = R_*$ we then obtain the differential equation

$$\frac{\partial^2 x}{\partial t^2} + (3\gamma - 4) \frac{3G\bar{\rho}}{4\pi} x(t) = 0$$

with

$$\bar{\rho} = \frac{M_*}{\frac{4\pi}{3} R_*^3}$$

Mean stellar density

with the usual solution

$$x(t) = x_0 e^{i\omega t}$$

Eigenfrequency of star

$$\omega^2 = (3\gamma - 4) \frac{3G\bar{\rho}}{4\pi}$$

eigenfrequency

$$P_{\text{puls}} \sim \sqrt{\frac{1}{\bar{\rho}}}$$

definition of pulsation period

of star
pulsation period:
strong radius dependence
weak mass dependence

\[ P_{\text{puls}} \sim \sqrt{\frac{1}{\rho}} \sim R_*^{\frac{3}{2}} M^{-\frac{1}{2}} \]

absolute magnitudes

\[ M_{V,I,J,H,K} \sim 5 \log R_* \]

predicted period luminosity relationship

\[ M_{V,I,J,H,K} \sim \frac{10}{3} \log P_{\text{puls}} \]


good agreement despite simplifications such as
adiabatic assumption

\[ P = P_0 \left( \frac{\rho}{\rho_0} \right)^\gamma \]

\[ \frac{\Delta r}{r_0} = x(t) \text{ independent of } r \]
PERIOD-LUMINOSITY RELATIONS

LMC

Period [days]

Magnitude (extinction corrected)

J

H

Ks

L. Macri, 2014, MIAPP WS

MACRI, NGEOW, KANBUR+ (IN PREP.)
WITHIN THE INSTABILITY STRIP

POP I

POP II

SX Phoenicis are found in GCs, dwarf galaxies, they trace Blue Straggler Stars or intermediate age pop (age~3-5Gyr), distance estimators?

Periods less than \( \leq 0.2 \) day.

RRLyrae stars trace the ancient stellar population (age\( \geq \)10Gyr), good distance estimators!!

Periods range from 0.2-1 day.
WITHIN THE INSTABILITY STRIP

**POP I**

Classical Cepheids are stellar tracers of young (age < 1 Gyr) populations, well known distance estimators!! Periods from few to ~80 days.

**POP II**

SX Phoenicis are found in GCs, dwarf galaxies, they trace Blue Straggler Stars or intermediate age pop (age ~3-5 Gyr), distance estimators? Periods less than ≤ 0.2 day.

RRLyrae stars trace the ancient stellar population (age ≥10 Gyr), good distance estimators!! Periods range from 0.2-1 day. Population II Cepheids, stars trace intermediate and ancient stellar population (age ≥10 Gyr), good distance estimators!! Periods >> 1 day.
**WITHIN THE INSTABILITY STRIP**

**POP I**
Classical Cepheids are stellar tracers of young (age < 1Gyr) populations, well known distance estimators!!

Periods from few to ~80 days

Ultra Long Period Cepheids are stellar tracers of very young (age << 50Myr) populations, distance estimators?

Periods > 80 days

**POP II**
SX Phoenicis are found in GCs, dwarf galaxies, they trace Blue Straggler Stars or intermediate age pop (age~3-5Gyr), distance estimators?

Periods less than ≤ 0.2 day.

RRLyrae stars trace the ancient stellar population (age≥10Gyr), good distance estimators!!

Periods range from 0.2-1 day.

Population II Cepheids, stars trace intermediate and ancient stellar population (age≥10Gyr), good distance estimators!!

Periods >> 1 day.
stellar stability

\[ \Delta r(t) = x(t) r_0 \]

time dependent radial movement of mass shells
described by oscillator equation, eigenfrequency
and adiabatic exponent \( \gamma \)

\[ \frac{\partial^2 x}{\partial t^2} + \omega^2 x(t) = 0 \]
\[ \omega^2 = (3\gamma - 4) \frac{3G\bar{\rho}}{4\pi} \]
\[ \gamma = \frac{c_p}{c_v} \]
\[ P = P_0 \left( \frac{\rho}{\rho_0} \right)^\gamma \]

periodic solution only for \( \gamma > \frac{4}{3} \) \( \iff \) \( \omega^2 > 0 \)

normal mono-atomic gas
non-relativistic Fermi gas \( \gamma = \frac{5}{3} \) \( \iff \) oscillation

however, for \( \gamma \leq \frac{4}{3} \) \( \iff \) \( \omega^2 \leq 0 \) \( \iff \) \( x(t) = x_0 e^{\pm |\omega|t} \)

exponential collapse or expansion

linear growth or collapse

\( \omega^2 = 0 \) \( x(t) = v_0 t \) instability !!!!
examples

• White Dwarf at Chandrasekhar-limit (Chapter 2)

• very massive stars dominated by radiation pressure

\[ P \approx P_{rad} = \frac{1}{3}aT^4 \]

with \( T = f(\rho(r)) \) from stellar interior structure \( \Rightarrow \) \( \gamma \sim \frac{4}{3} \)