Solve the problems listed below, and write up your answers clearly and completely. Do not turn in rough work – instead, make a clean copy after checking your calculations. Use English sentences and phrases to explain your solution and describe key equations. Show your work!

1. Consider a homogeneous planet with constant internal density $\rho$ and radius $R$.
(a) Derive an equation for the gravitational acceleration $g(r)$ at radii $r \leq R$.
(b) Use the equation of hydrostatic equilibrium,
$$\frac{dP}{dr} = -\rho g(r), \quad (1)$$
with boundary condition $P(R) = 0$, to derive an equation for $P(r)$.
(c) Use this equation to estimate the central pressure, $P_c = P(0)$, for Jupiter (radius $R_J = 7.1 \times 10^7$ m, average density $\rho_{\text{avg}} = 1.33 \times 10^3$ kg m$^{-3}$).

2. A better model for a gas giant planet, as discussed in class, adopts a “polytropic” equation of state,
$$P = K \rho^2. \quad (2)$$
Here $K$ is a constant with units of kg$^{-1}$ m$^5$ s$^{-2}$. The solution to the model equations is the density profile
$$\rho(r) = \rho_c \frac{\sin(\pi r/R)}{(\pi r/R)^2}, \quad (3)$$
where $\rho_c = (\pi^2/3)\rho_{\text{avg}}$ is the mass density at the center and $R = \sqrt{\pi K/(2G)}$ is the outer radius of the planet.
(a) Given Jupiter’s radius and average density from question #1, what are the corresponding values for $K$ and $\rho_c$? (Note: these values “fit” the model to Jupiter.)
(b) Using the values of $K$ and $\rho_c$ you obtained in part (a), what central pressure $P_c$ does this model predict?
(c) Compare your result from part (b) to the central pressure you obtained in question #1(c). Which model yields a higher pressure, and why?

3. Saturn radiates $L_{\text{rad}} \approx 1.98 \times 10^{17}$ kg m$^2$ s$^{-3}$ of infrared radiation, even though it absorbs only $W_{\text{sol}} \approx 1.11 \times 10^{17}$ kg m$^2$ s$^{-3}$ of solar radiation. Suppose that the difference is entirely provided by gravitational energy released by gradual contraction. Saturn’s mass is $M_S \approx 5.7 \times 10^{26}$ kg, and its current radius is $R_S \approx 5.8 \times 10^7$ m. To calculate the gravitational energy, you may assume that Saturn has uniform density, and use the formula appropriate for a homogeneous planet.
(a) How fast is Saturn’s radius changing; that is, what is $dR_S/dt$?
(b) Assuming it continued contracting at this rate for $10^9$ yr, by what percentage would it have shrunk?

4. Saturn’s average density $\bar{\rho} \simeq 690 \text{ kg m}^{-3}$ is less than the density of water. Elementary astronomy textbooks sometimes dramatize this fact by suggesting that Saturn would float in a (very big) tub of water. Is this at all plausible? What would happen if you tried?