Solve the problems listed below, and write up your answers clearly and completely. Do not turn in a rough draft – make a clean copy after checking your calculations. Use brief sentences and phrases to explain your solution and introduce symbols and key equations. Show your work!

1. Consider a homogeneous planet with constant internal density $\rho$ and radius $R$.
   (a) Derive an equation for the gravitational acceleration $g(r)$ at radii $r \leq R$.
   (b) Use the equation of hydrostatic equilibrium,
   
   $$\frac{dP}{dr} = -\rho g(r), \quad (1)$$

   with boundary condition $P(R) = 0$, to derive an equation for $P(r)$.
   (c) Use this equation to estimate the central pressure, $P_c = P(0)$, for Jupiter (radius $R_J = 7.1 \times 10^7$ m, average density $\rho_{avg} = 1.33 \times 10^3$ kg m$^{-3}$).

2. A better model for a gas giant planet, as discussed in class, adopts a “polytropic” equation of state,

   $$P = K\rho^2. \quad (2)$$

   Here $K$ is a constant with units of kg$^{-1}$ m$^5$ s$^{-2}$. The solution to the model equations is the density profile

   $$\rho(r) = \rho_c \frac{\sin(\pi r/R)}{(\pi r/R)}, \quad (3)$$

   where $\rho_c = (\pi^2/3)\rho_{avg}$ is the mass density at the center and $R = \sqrt{\pi K/(2G)}$ is the outer radius of the planet.
   (a) Given Jupiter’s radius and average density from question #1, what are the corresponding values for $K$ and $\rho_c$? (Note: these values “fit” the model to Jupiter.)
   (b) Using the values of $K$ and $\rho_c$ you obtained in part (a), what central pressure $P_c$ does this model predict?
   (c) Compare your result from part (b) to the central pressure you obtained in question #1(c). Which model yields a higher pressure, and why?

3. Saturn radiates $L_{rad} \simeq 1.98 \times 10^{17}$ kg m$^2$ s$^{-3}$ of infrared radiation, even though it absorbs only $W_{sol} \simeq 1.11 \times 10^{17}$ kg m$^2$ s$^{-3}$ of solar radiation. Suppose that the difference is entirely provided by gravitational energy released by gradual contraction. Saturn’s mass is $M_S \simeq 5.7 \times 10^{26}$ kg, and its current radius is $R_S \simeq 5.8 \times 10^7$ m. To calculate the gravitational energy, you may assume that Saturn has uniform density, and use the formula appropriate for a homogeneous planet.
   (a) How fast is Saturn’s radius changing; that is, what is $dR_S/dt$?
   (b) Assuming it continued contracting at this rate for $10^9$ yr, by what percentage would it have shrunk?