1. (a) Newton's "Iron Sphere" theorems show that the mass outside radius $r$ has no effect, and the mass within $r$ acts as if concentrated at the center.

The mass inside radius $r \leq R$ is

$$M(r) = \frac{4\pi}{3} r^3 \rho,$$

so the acceleration at $r \leq R$ is

$$g(r) = \frac{GM(r)}{r^2} = \frac{4\pi}{3} G \rho r.$$

(b) With this $g(r)$, the HSE equation becomes

$$dP = -\frac{4\pi}{3} G \rho^2 r \, dr.$$

To get the pressure $P(r)$, integrate from $r$ to the surface, since pressure is weight of material above $r$:

$$\int P(R) \, dp = -\frac{4\pi}{3} G \rho^2 \int_r^R r' \, dr'.$$

$$P(R) - P(r) = -\frac{2\pi}{3} G \rho^2 (R^2 - r^2),$$

$$\therefore \ P(r) = \frac{2\pi}{3} G \rho^2 (R^2 - r^2).$$

[using $P(R) = 0$]

In terms of $M$ instead of $\rho$, the pressure is

$$P(r) = \frac{2}{8\pi} G \frac{M^2}{R^6} (R^2 - r^2).$$

(c) The central pressure is

$$P_c = P(0) = \frac{2\pi}{3} G \rho^2 \frac{R^5}{R^6} = 1.25 \times 10^5 \text{ kg m}^{-1} \text{s}^{-2}.$$

Units: $[G \rho^2 R^2] = \text{kg} \cdot \text{m}^3 \text{s}^{-2} \cdot \text{kg}^2 \cdot \text{m}^{-6} \cdot \text{m}^2 = \text{kg} \cdot \text{m}^{-1} \text{s}^{-2} \checkmark$

(more)
2. (a) Given values for $R$ and $p_m$, solve for the model parameters $K$ and $p_e$:

$$K = \frac{2}{\pi} G R^2 = 2.15 \times 10^5 \text{ kg m}^{-1} \text{ s}^{-2}$$

$$p_e = \frac{\pi^2}{3} p_m = 4.4 \times 10^3 \text{ kg m}^{-3}$$

A gas sphere model with these parameters will have the same radius and mass as Jupiter, so it "fits".

(b) The central pressure follows by plugging the central density $p_e$ into the equation of state:

$$p_e = K p_e^2 = (2.15 \times 10^5 \text{ kg m}^{-1} \text{ s}^{-2}) (4.4 \times 10^3 \text{ kg m}^{-3})^2$$

$$= 4.16 \times 10^{12} \text{ kg m}^{-1} \text{ s}^{-2}$$

(c) The gas sphere model yields a pressure more than 3 times higher than the homogeneous model. This is because the concentration of mass toward the center creates a stronger gravitational field at $r < R$, increasing the weight of the overlying gas.

![Graph showing pressure vs radius for two models of Jupiter](image)

Pressure vs radius for two models of Jupiter (more)
3. (a) When a gas planet contracts, half the gravitational energy released is needed to increase molecular velocity in the interior, and half is available to power escaping thermal radiation. So the net power required is

\[ W_{\text{grav}} = 2 (L_{\text{rad}} - W_{\text{sel}}) = 1.74 \times 10^{17} \text{ J m}^{-2} \text{ s}^{-3} \]

According to the homogeneous model, \( E_{\text{grav}} = \frac{3}{5} G \frac{M^2}{R} \), so

\[ dE_{\text{grav}} = -\frac{3}{5} G \frac{M^2}{R^2} dR \]

Now \( W_{\text{grav}} = \frac{dE_{\text{grav}}}{dt} \), so

\[ W_{\text{grav}} \frac{dt}{dR} = -\frac{3}{5} G \frac{M^2}{R^2} \]

Reomgining to get \( dR/dt \):

\[ \frac{dR}{dt} = -\frac{5}{3} \frac{R^2}{M^2} W_{\text{grav}} = 4.5 \times 10^{-11} \text{ m s}^{-1} \]

(b) At this (slow) rate, in \( 10^9 \text{ yr} = 3.15 \times 10^{16} \text{ s} \), Saturn would contract by

\[ \Delta R_s = \frac{dR}{dt} \cdot 10^9 \text{ yr} = 1.41 \times 10^6 \text{ m} \]

In terms of Saturn's radius \( R_s = 5.8 \times 10^7 \text{ m} \), this is

\[ \Delta R_s / R_s \approx 0.024 \quad \text{or} \quad 2.4\% \quad \text{per} \quad 10^9 \text{ yr} \]

This is an upper limit on Saturn's actual rate of contraction. For an in-homogeneous planet, \( E_{\text{grav}} > \frac{3}{5} G \frac{M^2}{R} \), because allowing denser regions to sink downward releases energy. Thus more grav. energy is available and the rate of contraction needed to supply \( W_{\text{grav}} \) is lower.