Astronomy 241: Problem Set #3

due September 12, 2016

Solve the problems listed below, and write up your answers clearly and completely. Do not turn in a rough draft – make a clean copy after checking your calculations. Use brief sentences and phrases to explain your solution and introduce symbols and key equations. Show your work!

1. A Hohmann transfer orbit, as described in Chapter 3.3, is an ellipse tangent to one orbit at perihelion, and tangent to a second orbit at aphelion (Fig. 1). Assume that the Earth and Jupiter revolve around the Sun on circular orbits with $a_E = 1$ AU and $a_J = 5$ AU. Solve the following problems:
   (a) Find the semi-major and semi-minor axes $a_{tr}$ and $b_{tr}$ of the transfer orbit.
   (b) Calculate the period $P_{tr}$ of this orbit. How long does the trip to Jupiter take?
   (c) Find the velocity of this orbit at perihelion and at aphelion.
   (d) If a spaceship is in a circular orbit around the Sun at $a = 1$ AU, what velocity increment $\Delta v$ is needed to place it in this transfer orbit?

2. For the one-body problem, the general expression for orbital radius $r$ as a function of angle $\theta$ is

$$r(\theta) = \frac{(L/m)^2}{GM(1 + e \cos \theta)}.$$  \hspace{1cm} (1)

This is valid for any eccentricity $e \geq 0$. In what follows, you should make sure that your results are also valid for any $e \geq 0$.
   (a) Find the value of $r$ at pericenter; this is the pericentric separation, written $r_p$.
   (b) Using $dA/dt = L/(2m)$, calculate the velocity at pericenter, or pericentric velocity, written $v_p$, and express the result in terms of $G$, $M$, $e$, and $r_p$.

Figure 1: A Hohmann transfer orbit (dotted line) between two circular orbits (solid lines).
(c) Substitute $r_p$ and $v_p$ into the expression for the total energy,

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} \quad (2)$$

and express the result in terms of $G$, $M$, $m$, $e$, and $r_p$.

(d) How does the sign of $E$ depend on the eccentricity $e$?

(e) Consider a body on a hyperbolic orbit with eccentricity $e = 2$. What speed will the body have as $r \to \infty$?

3. A satellite of mass $m$ is on a circular orbit of radius $r$ around a planet of mass $M \gg m$. Suppose that we instantaneously increase the satellite’s speed by a factor of $\alpha \geq 1$ (that is, set $\vec{v}_{\text{new}} = \alpha \vec{v}_{\text{old}}$).

(a) Compute the major axis, minor axis, pericenter distance, and apocenter distance of the new orbit.

(b) Note that that $r_{\text{apo}} \to \infty$ as $\alpha \to \sqrt{2}$. Explain this result by applying the virial theorem ($2K + U = 0$) to the initial orbit.