Chapter 7

Gravitational Fields

Analytic calculation of gravitational fields is easy for spherical systems, but few *general* results are available for non-spherical mass distributions. Mass models with tractable potentials may however be combined to approximate the potentials of real galaxies.

7.1 Conservative Force Fields

In a one-dimensional system it is always possible to define a potential energy corresponding to any given $f(x)$; let

$$U(x) = -\int_{x_0}^{x} dx' f(x') \,, \tag{7.1}$$

where x_0 is an arbitrary position at which $U = 0$. Different choices of x_0 produce potential energies differing by an additive constant; this constant has no influence on the dynamics of the system.

In a space of $n > 1$ dimensions the analogous path integral,

$$U(\mathbf{x}) = -\int_{\mathbf{x}_0}^{\mathbf{x}} d^n\mathbf{x}' \cdot \mathbf{f}(\mathbf{x}') \,, \tag{7.2}$$

may depend on the exact route taken from points $\mathbf{x}_0$ to $\mathbf{x}$; if it does, a unique potential energy cannot be defined. One condition for this integral to be path-independent is that the integral of the force $\mathbf{f}(\mathbf{x})$ around all closed paths vanishes. An equivalent condition is that there is some function $U(\mathbf{x})$ such that

$$\mathbf{f}(\mathbf{x}) = -\nabla U \,. \tag{7.3}$$

Force fields obeying these conditions are *conservative*. The gravitational field of a stationary point mass is an example of a conservative field; the energy released in moving from radius r_1 to radius $r_2 < r_1$ is *exactly* equal to that consumed in moving back from r_2 to r_1.

7.2 General Results

In astrophysical applications it's natural to work with the path integral of the *acceleration* rather than the force; this integral is the potential energy per unit mass or **gravitational potential**, $\Phi(\mathbf{x})$, and the potential energy of a test mass m is just $U = m\Phi(\mathbf{x})$. For an arbitrary mass density $\rho(\mathbf{x})$, the potential is

$$\Phi(\mathbf{x}) = -G \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \tag{7.4}$$

where G is the gravitational constant and the integral is taken over all space.

Poisson's equation provides another way to express the relationship between density and potential:

$$\nabla^2\Phi = 4\pi G\rho\,. \tag{7.5}$$

Note that these relationships are *linear*; if ρ_1 generates Φ_1 and ρ_2 generates Φ_2 then $\rho_1+\rho_2$ generates $\Phi_1+\Phi_2$.

Gauss's theorem relates the mass within some volume V to the gradient of the field on its surface dV:

$$4\pi G \int_V d^3\mathbf{x}\rho(\mathbf{x}) = \int_{dV} d^2\mathbf{S}\cdot\nabla\Phi\,, \tag{7.6}$$

where $d^2\mathbf{S}$ is an element of surface area with an outward-pointing normal vector.

7.3 Spherical Potentials

Consider a spherical shell of mass m; Newton's first and second theorems (BT87, Ch. 2.1) imply

- the acceleration inside the shell vanishes, and
- the acceleration outside the shell is $-Gm/r^2$.

From this, it follows that the potential of an *arbitrary* spherical mass distribution is

$$\Phi(r) = -\int_{r_0}^{r} dx\, a(x) = G\int_{r_0}^{r} dx\,\frac{M(x)}{x^2}\,, \tag{7.7}$$

where the enclosed mass is

$$M(r) = 4\pi\int_0^r dx\, x^2\rho(x)\,. \tag{7.8}$$

If M converges for $r\to\infty$, it's convenient to set $r_0=\infty$; then $\Phi(r)<0$ for all finite r.

7.3.1 Elementary Examples

A **point** of mass M has the potential

$$\Phi(r) = -G\frac{M}{r}\,. \tag{7.9}$$

This is known as a *Keplerian* potential since orbits in this potential obey Kepler's three laws. A circular orbit at radius r has velocity $v_\mathrm{c}(r) = \sqrt{GM/r}$.

A **uniform sphere** of mass M and radius a has the potential

$$\Phi(r) = \begin{cases} -2\pi G\rho(a^2 - r^2/3)\,, & r<a \\ -GM/r\,, & r>a \end{cases} \tag{7.10}$$

where $\rho = M/(4\pi a^3/3)$ is the mass density. Outside the sphere the potential is Keplerian, while inside it has the form of a parabola; both the potential and its derivative are continuous at the surface of the sphere.

A **singular isothermal sphere** with density profile $\rho(r) = \rho_0(r/r_0)^{-2}$ has the potential

$$\Phi(r) = 4\pi G\rho_0 r_0^2 \ln(r/r_0)\,. \tag{7.11}$$

The circular velocity $v_c = \sqrt{4\pi G\rho_0 r_0^2}$ is constant with radius. This potential is often used to approximate the potentials of galaxies with flat rotation curves, but some outer cut-off must be imposed to obtain a finite total mass.

7.3.2 Potential-Density Pairs

Pairs of functions related by Poisson's equation provide convenient building-blocks for galaxy models. Three such functions often used in the literature are listed here; all describe models characterized by a total mass M and a length scale a:

Name	$\rho(r)$	$\Phi(r)$
Plummer	$\frac{3M}{4\pi a^3}\left(1+\frac{r^2}{a^2}\right)^{-5/2}$	$\frac{-GM}{\sqrt{r^2+a^2}}$
Hernquist	$\frac{M}{2\pi}\frac{a}{r(r+a)^3}$	$\frac{-GM}{r+a}$
Jaffe	$\frac{M}{4\pi}\frac{a}{r^2(r+a)^2}$	$\frac{GM}{a}\ln\left(\frac{a}{r+a}\right)$
Gamma	$\frac{(3-\gamma)M}{4\pi a^3}\frac{a^4}{r^\gamma(r+a)^{4-\gamma}}$	$\frac{GM}{a}\begin{cases}\frac{1}{\gamma-2}\left[1-\left(\frac{r}{r+a}\right)^{2-\gamma}\right], & \gamma\neq 2\\ \ln\left(\frac{r}{r+a}\right), & \gamma=2\end{cases}$

The Plummer (1911) density profile has a finite-density core and falls off as r^{-5} at large radii; this is a steeper fall-off than is generally seen in galaxies. Hernquist (1990) and Jaffe (1983) models, on the other hand, both decline like r^{-4} at large radii; this power law has a sound theoretical basis in the mechanics of violent relaxation. The Hernquist model has a gentle power-law cusp at small radii, while the Jaffe model has a steeper cusp. Gamma models (Dehnen 1993; Tremaine et al. 1994) include both Hernquist ($\gamma = 1$) and Jaffe ($\gamma = 2$) models as special cases; the best approximation to the de Vaucouleurs profile has $\gamma = 3/2$.

7.4 Axisymmetric Potentials

If the mass distribution is a function of two variables, cylindrical radius R and height z, the problem of calculating the potential becomes a good deal harder. A general expression exists for infinitely thin disks, but only special cases are known for systems with finite thickness.

7.4.1 Thin disks

An axisymmetric disk is described by a surface mass density $\Sigma(R)$. Potential-surface density expressions for several important cases are collected here:

Name	$\Sigma(R)$	$\Phi(R,z)$
Kuzmin	$\frac{aM}{2\pi(R^2+a^2)^{3/2}}$	$\frac{-GM}{\sqrt{R^2+(a+\lvert z\rvert)^2}}$
Toomre	$\left(\frac{d}{da^2}\right)^{n-1}\Sigma_{\mathrm{Kuzmin}}$	$\left(\frac{d}{da^2}\right)^{n-1}\Phi_{\mathrm{Kuzmin}}$
Bessel	$\frac{k}{2\pi G}J_0(kR)$	$\exp(-k\lvert z\rvert)J_0(kR)$

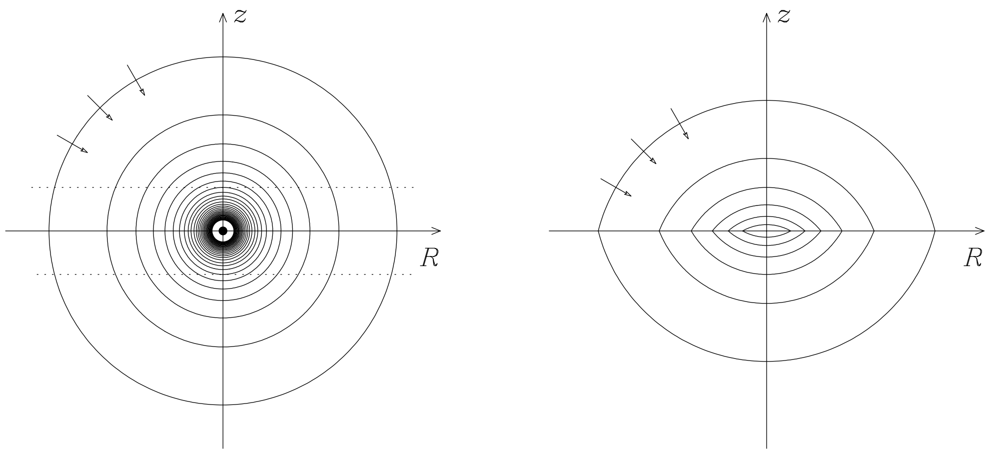

Figure 7.1: Surgery on the potential of a point mass produces the potential of a Kuzmin disk. Left: potential of a point mass. Contours show equal steps of $\Phi \propto 1/r$, while the arrows in the upper left quadrant show the radial force field. Dotted lines show $z = \pm a$. Right: potential of a Kuzmin disk, produced by excising the region $|z| \leq a$ from the field shown on the left. Arrows again indicate the force field; note that these no longer converge on the origin.

The potential of a Kuzmin (1956) disk may be constructed by performing surgery on the field of a point of mass M (7.9), as shown in Fig. 7.1. Let the z coordinate be the axis of the disk, and join the field for $z > a > 0$ directly to the field for $z < -a$. The result satisfies Laplace's equation, $\nabla^2\Phi = 0$, everywhere $z \neq 0$; thus the matter density which generates this potential is confined to the disk plane $z = 0$. Gauss's theorem (7.6) can be used to find the required surface density. Consider an infinitesimal box which straddles the disk plane at radius R, and let the top and bottom surfaces of this box have area δA. The integral of $\nabla\Phi$ over the surface of the box is $2GMa\delta A/(R^2 + a^2)^{3/2}$, and mass within the box is $\delta A \Sigma(R)$. Setting these two integrals equal yields the surface density $\Sigma_{\rm K}$ of Kuzmin's disk.

Toomre (1962) devised an infinite sequence of models, parameterized by the integer n. The $n = 1$ Toomre model is identical to Kuzmin's disk, while the $n = 2$ model is found by differentiating the potential and surface density of Kuzmin's disk by the parameter a^2; the linearity of Poisson's equation (7.5) guarantees that the result is a valid potential-surface density pair. In the limit $n \to \infty$ the surface density is a gaussian in R.

The 'Bessel' potential-density pair, also due to Toomre (1962), describes another field which obeys Laplace's equation everywhere except on the disk plane. Unlike other cases, the surface density – once again obtained using Gauss's theorem – is not positive-definite. What good is it? A disk with arbitrary surface density $\Sigma(R)$ may be expanded as

$$\Sigma(R) = \int dk\, S(k) k J_0(kR) \tag{7.12}$$

where $S(k)$ is the Hankel transform of $\Sigma(R)$,

$$S(k) = \int dR\, \Sigma(R) R J_0(kR)\,. \tag{7.13}$$

The potential generated by this disk is then

$$\Phi(R,z) = -2\pi G \int dk \exp(-k|z|) J_0(kR) S(k) \,. \tag{7.14}$$

This method may be used to evaluate the potential of an exponential disk. Consider a disk with surface density

$$\Sigma(R) = \frac{\alpha^2 M}{2\pi} e^{-\alpha R} = \Sigma_0 e^{-\alpha R} \,, \tag{7.15}$$

where M is the total disk mass and α is the inverse of the disk scale length. The resulting potential is

$$\Phi(R,z) = -\frac{2\pi G \Sigma_0}{\alpha^2} \int_0^\infty dk \frac{J_0(kR) e^{-k|z|}}{[1 + k^2 \alpha^{-2}]} \,. \tag{7.16}$$

In the plane of the disk, $z = 0$ and this integral becomes

$$\Phi(R,0) = -\pi G \Sigma_0 R [I_0(y) K_1(y) - I_1(y) K_0(y)] \,, \tag{7.17}$$

where $y = \alpha R/2$, and I_n and K_n are modified Bessel functions of the first and second kind.

7.4.2 Flattened Systems

Potential-density pairs for a few flattened systems are listed here:

Name	$\rho(R,z)$	$\Phi(R,z)$
Miyamoto-Nagai	$\left(\frac{b^2 M}{4\pi}\right) \frac{aR^2 + (a + 3\sqrt{z^2+b^2})(a+\sqrt{z^2+b^2})}{[R^2 + (a+\sqrt{z^2+b^2})^2]^{5/2} (z^2+b^2)^{3/2}}$	$\frac{-GM}{\sqrt{R^2 + (a+\sqrt{b^2+z^2})^2}}$
Satoh	$\left(\frac{d}{db^2}\right)^n \Phi_{\mathrm{MN}}$	$\left(\frac{d}{db^2}\right)^n \Phi_{\mathrm{MN}}$
Logarithmic	$\left(\frac{v_0^2}{4\pi G q^2}\right) \frac{(2q^2+1)R_c^2 + R^2 + (2-q^{-2})z^2}{(R_c^2 + R^2 + z^2 q^{-2})^2}$	$\frac{1}{2} v_0^2 \ln(R_c^2 + R^2 + z^2/q^2)$

The Miyamoto & Nagai (1975) potential includes the Plummer model (if $a = 0$) and the Kuzmin disk (if $b = 0$); thus it can describe a wide range of shapes, from a spherical system to an infinitely thin disk. Satoh's (1980) models are derived from the Miyamoto-Nagai model by differentiating with respect to the parameter b^2, much as Toomre's models are derived from Kuzmin's.

The logarithmic potential is often used to describe galaxies with approximately flat rotation curves; in the $z = 0$ plane, the circular velocity $v_c = v_0 R/\sqrt{R_c^2 + R^2}$ rises linearly for $R \ll R_c$ and levels off for large R. The corresponding density can be found by inserting the potential in Poisson's equation. As Fig. 2.8 of BT87 shows, this density is 'dimpled' at the poles; if $q^2 < 1/2$ the density must actually be *negative* along the z axis, which is unphysical. This dimpling is demanded by the assumption that $\Phi(R,z)$ is stratified on surfaces of constant ellipticity as $R \to \infty$. In models where the potential becomes more spherical at larger radii this dimpling isn't necessary.

A great deal of analytic machinery exists to calculate potentials for systems with densities stratified on ellipsoidal surfaces (BT87, Ch. 2.3). Much of this machinery generalizes to triaxial systems; a few key results are given below.

7.5 Triaxial Potentials

7.5.1 Ellipsoidal systems

The triaxial generalization of an infinitely thin spherical shell is an infinitely thin *homoeoid*, which has constant density between surfaces m^2 and $m^2 + dm^2$, where

$$m^2 = x^2 + \frac{y^2}{b^2} + \frac{z^2}{c^2} . \tag{7.18}$$

Just as in the spherical case, the acceleration inside the shell vanishes; thus $\Phi =$ const. inside the shell and on its surface. Outside the shell, the potential is stratified on ellipsoidal surfaces defined by

$$m^2 = \frac{x^2}{1+\tau} + \frac{y^2}{b^2+\tau} + \frac{z^2}{c^2+\tau} \tag{7.19}$$

where the parameter $\tau > 0$ labels the surface. Notice that for $\tau = 0$ the isopotential surface coincides with the surface of the homoeoid, while in the limit $\tau \to \infty$ the isopotential surfaces become spherical.

A triaxial mass distribution with $\rho = \rho(m^2)$ is equivalent to the superposition of a series of thin homoeoids. The acceleration at a given point (x_0, y_0, z_o) is generated entirely by the mass within $m^2 < m_0^2 = x_0^2 + y_0^2/b^2 + z_0^2/c^2$, just as in a spherical system the acceleration at r is due only to the mass within r. Since the isopotential surfaces of a homoeoid become spherical at large r, mass lying within $m^2 \ll m_0$ generates a nearly radial force field, while mass lying only slightly within m_0^2 generates a more non-radial force field. Thus the potential of a centrally concentrated body is always more spherical than the potential of uniform body with the same axial ratios.

7.5.2 Logarithmic potential

The triaxial version of the logarithmic potential is

$$\Phi(x,y,z) = \frac{1}{2} v_0^2 \ln \left(R_c^2 + x^2 + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) . \tag{7.20}$$

As in the 2-D case above, the corresponding density is not guaranteed to be positive if the potential is too strongly flattened. Nonetheless, the logarithmic potential is a useful approximation when studying orbits in triaxial galaxies.

Problems

7.1. Evil space aliens compress the Earth to an infinitely thin disk of uniform surface density $\Sigma_\oplus$ without changing its radius. What is the gravitational acceleration experienced by a survivor standing upright at the center of this disk?

7.2. Figure 2-17 of BT87 compares circular-speed curves $v_c(r)$ for three mass distributions: "an exponential disk (full curve); a point with the same total mass (dotted curve); the spherical body for which $M(r)$ is [identical to $M(r)$ for the exponential disk] (dashed curve)" (see BT87, p. 78). In qualitative terms, first explain why the dashed curve lies *below* the dotted curve at all radii. Then, explain why the full curve lies *above* the dotted curve at *some* radii.