Chapter 11

(How to) Not Solve the Collisionless Boltzmann Equation

The collisionless Boltzmann equation (CBE) is a partial differential equation obeyed by a function of six dimensions. We can tame this unwieldy equation by integrating along one or more dimensions; each integration literally reduces the CBE to a shadow of its former self, but for some purposes these shadows contain all the information needed to solve a particular problem.

This and subesequent chapters will sometimes use the Einstein summation convention, in which repeated indicies in a product are summed over: $\mathbf{a} \cdot \mathbf{b} = a_i b_i = \sum_{i=1}^{3} a_i b_i$. In this notation, the CBE is

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial r_i} - \frac{\partial \Phi}{\partial r_i} \frac{\partial f}{\partial v_i} = 0 . \tag{11.1}$$

11.1 The Jeans Equations

To begin with, we can project out the velocity dimensions. There are many ways to do this (BT08, Chapter 4.8); the first of these is simply to integrate over all velocities:

$$\int d\mathbf{v} \, \frac{\partial f}{\partial t} + \int d\mathbf{v} \, v_i \frac{\partial f}{\partial r_i} - \int d\mathbf{v} \, \frac{\partial \Phi}{\partial r_i} \frac{\partial f}{\partial v_i} = 0 . \tag{11.2}$$

Rearranging the first two terms, and using the divergence theorem (BT08, Appendix B.3) to show that the third term vanishes, we get a continuity equation for the 3-D stellar density,

$$\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial r_i} (\nu \overline{v_i}) = 0 , \tag{11.3}$$

where the stellar mass density and momentum density are

$$\nu \equiv \int d\mathbf{v} \, f , \qquad \nu \overline{v_i} \equiv \int d\mathbf{v} \, f v_i , \tag{11.4}$$

with the integrals taken over all velocities.

To obtain the next (three) equations in this series, multiply the CBE by the velocity component v_j, and again integrate over all velocities. Proceeding much as before, the result is

$$\frac{\partial}{\partial t} (\nu \overline{v_j}) + \frac{\partial}{\partial r_i} (\nu \overline{v_j v_i}) + \nu \frac{\partial \Phi}{\partial r_j} = 0 . \tag{11.5}$$

where

$$\nu\overline{v_j v_i} \equiv \int d\mathbf{v}\, f v_j v_i \,. \tag{11.6}$$

This may be placed in a 'more familiar' form by defining the dispersion tensor

$$\sigma_{ij}^2 \equiv \overline{v_j v_i} - \overline{v_j}\,\overline{v_i} \tag{11.7}$$

which represents the distribution of stellar velocities with respect to the mean at each point. The result is sometimes called the equation of stellar hydrodynamics,

$$\nu\frac{\partial \overline{v_j}}{\partial t} + \nu\overline{v_i}\frac{\partial \overline{v_j}}{\partial r_i} = -\nu\frac{\partial \Phi}{\partial r_j} - \frac{\partial}{\partial r_i}(\nu\sigma_{ij}^2)\,, \tag{11.8}$$

because it resembles Euler's equation of fluid flow, with the last term on the right representing an anisotropic pressure. Note that because there is no equation of state, this pressure is not related in any simple way to the mass and momentum density defined in (11.4).

By multiplying the CBE by $v_i v_j ... v_l$ and integrating over all velocities, yet higher order equations can be formed; each, alas, makes reference to quantities of one order higher than those it describes. To close this hierarchy of equations, one must make some physical assumptions.

Caveat: Any *physical* solution of the CBE must have a non-negative phase-space distribution function. The velocity moments of the CBE describe possible solutions, but do *not* guarantee that a non-negative $f(r,v;t)$ exists. You've been warned!

11.2 The Tensor Virial Theorem

To derive the tensor virial equation, we multiply the CBE by $v_j r_k$ and integrate over all velocities *and* positions (BT08, Chapter 4.8.3). First, we multiply by v_j and integrate over velocity; this yields (11.5). Next, multiplying each term by r_k and integrating over position, we get

$$\frac{\partial}{\partial t}\int d\mathbf{r}\, r_k \nu \overline{v_j} = -\int d\mathbf{r}\, r_k \frac{\partial}{\partial r_i}(\nu\overline{v_j v_i}) - \int d\mathbf{r}\, r_k \nu\frac{\partial \Phi}{\partial r_j}\,, \tag{11.9}$$

where the time derivative has been moved outside the integral on the LHS. We will come back to this term shortly, but it is worth noting that for a system in equilibrium the integral is constant, and thus the LHS is zero. The first term on the RHS may be simplified using the divergence theorem:

$$-\int d\mathbf{r}\, r_k \frac{\partial}{\partial r_i}(\nu\overline{v_j v_i}) = \int d\mathbf{r}\, \nu\overline{v_j v_k} = 2K_{jk}\,. \tag{11.10}$$

Here K_{jk} is the *kinetic energy tensor*, defined as

$$K_{jk} \equiv \frac{1}{2}\int d\mathbf{r} d\mathbf{v}\, f(\mathbf{r},\mathbf{v}) v_j v_k \tag{11.11}$$

Note that K_{jk} is symmetric, $K_{jk} = K_{kj}$, and that its trace is the total kinetic energy of the system.

The second term on the RHS is the *potential energy tensor*,

$$W_{jk} \equiv -\int d\mathbf{r}\, r_k \nu\frac{\partial \Phi}{\partial r_j}\,. \tag{11.12}$$

Under the assumption that the stellar mass density $\nu(\mathbf{r})$ is also the source of the gravitational field, it follows that W_{jk} is symmetric, that its trace is the potential energy U, that for a spherical system

$W_{jk} = \delta_{jk}U/3$, and that for a system flattened along the z direction $W_{xx}/W_{zz} > 1$ (BT08, Chapter 2.1).

Using the symmetry of K_{jk} and W_{jk}, (11.9) becomes

$$\frac{1}{2}\frac{d}{dt}\int d\mathbf{r}\,\nu(r_k\overline{v_j} + r_j\overline{v_k}) = 2K_{jk} + W_{jk}\,. \tag{11.13}$$

The LHS, explicitly symmetrized over k and j, is one-half of the second time derivative of the *moment of inertia tensor*,

$$I_{jk} = \int d\mathbf{r}\,\nu r_j r_k\,. \tag{11.14}$$

Putting everything together finally gives the tensor virial equation,

$$\frac{1}{2}\frac{d^2}{dt^2}I_{jk} = 2K_{jk} + W_{jk}\,. \tag{11.15}$$

Remark: the trace of (11.15) is the more familiar scalar virial theorem; see BT08, Chapter 4.8.3(a) for a discussion.