N-Body Methods: Review and Topics

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YITP, Kyoto University, 1 June 2015
Stellar Dynamics

Working model of a galaxy: stars and/or particles of dark matter.

\[ \{ (m_i, r_i, v_i) \mid i = 1, \ldots, N \} \]

Evolution is described by the self-consistent N-body equations:

\[
\frac{dr_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \sum_{j \neq i}^N \frac{G m_j (r_j - r_i)}{|r_j - r_i|^3}
\]

The Lagrangian is \( L(r_i, v_i, t) = K - W \), where \( v \equiv \dot{r} \) and

\[
K \equiv \frac{1}{2} \sum_i^N m_i v_i^2, \quad W \equiv -\frac{1}{2} \sum_i^N \sum_{j \neq i}^N \frac{G m_i m_j}{|r_j - r_i|}
\]

Symmetries of \( L \) imply conservation of energy \( E = K + W \), momentum, and angular momentum.
General Results.

Virial Theorem: \[ \frac{1}{2} \frac{d^2 I}{dt^2} = 2K + W \]

\[ I \equiv \sum_i^N m_i \mathbf{r}_i \otimes \mathbf{r}_i \quad \quad K \equiv \frac{1}{2} \sum_i^N m_i \mathbf{v}_i \otimes \mathbf{v}_i \]

\[ W \equiv -\frac{1}{2} \sum_i^N \sum_{j\neq i}^N \frac{G m_i m_j (\mathbf{r}_i-\mathbf{r}_j) \otimes (\mathbf{r}_i-\mathbf{r}_j)}{|\mathbf{r}_i-\mathbf{r}_j|^3} \]

Use center-of-mass frame, assume equilibrium, and take trace:

\[ 0 = \frac{1}{2} \frac{d^2 I}{dt^2} = 2K + W \quad \text{or} \quad E = \frac{1}{2} W = -K \]

Define virial velocity, radius, and crossing time (\( M \equiv \text{total mass} \)):

\[ \mathcal{V} \equiv \sqrt{-2E / M} \quad \mathcal{R} \equiv -GM^2 / 2E \quad t_c \equiv \mathcal{R} / \mathcal{V} \]
General Results. II

State of N-body system is point in $6N$ dimensional $\Gamma$–space:

$$(r_1, v_1, \ldots, r_N, v_N)$$

$f^{(N)}(r_1, \ldots, v_N, t) \equiv$ density of systems in $\Gamma$–space (e.g., ensemble of equivalent micro-states).

Liouville’s Theorem: flow of points in $\Gamma$–space is *incompressible*:

$$\frac{df^{(N)}}{dt} = \frac{\partial f^{(N)}}{\partial t} + \sum_{i=1}^{N} \left( v_i \cdot \frac{\partial f^{(N)}}{\partial r_i} - \frac{\partial \Phi_i}{\partial r_i} \cdot \frac{\partial f^{(N)}}{\partial v_i} \right) = 0$$

It’s convenient to assume that $f^{(N)}$ is symmetric (in other words, bodies are interchangeable).
Vlasov Equation: Derivation

Define 1-body and 2-body distribution functions:

\[ f(r_1, v_1, t) \equiv \int f^{(N)}(r_1, ..., v_N, t) \, d^3r_2 \ldots d^3v_N \]

\[ f^{(2)}(r_1, v_1, r_2, v_2, t) \equiv \int f^{(N)}(r_1, ..., v_N, t) \, d^3r_3 \ldots d^3v_N \]

Integrate Liouville’s equation over \( d^3r_2 \ldots d^3v_N \) to get:

\[
\frac{\partial f}{\partial t} + v_1 \cdot \frac{\partial f}{\partial r_1} - (N - 1) \int \frac{\partial \Phi_{12}}{\partial r_1} \cdot \frac{\partial f^{(2)}}{\partial v_1} \, d^3r_2 \, d^3v_2 = 0
\]

Assume bodies aren’t correlated: \( f^{(2)}(1, 2, t) = f(1, t) f(2, t) \)

\[
\frac{\partial f}{\partial t} + v_1 \cdot \frac{\partial f}{\partial r_1} - \left( \frac{N - 1}{N} \right) \frac{\partial \Phi}{\partial r_1} \cdot \frac{\partial f}{\partial v_1} = 0
\]
2-Body Relaxation

In single encounter, body acquires velocity

\[ \delta v_t \sim \frac{2Gm}{bv}, \quad b \gg b_{\text{min}} \equiv \frac{Gm}{v^2} \approx \frac{R}{N} \]

Uncorrelated encounters \( \Rightarrow \) random walk in velocity; change each crossing due to encounters with impact parameter \( b \) to \( b + db \) is

\[ d(v^2) = (\delta v_t)^2 dn \approx \left( \frac{2Gm}{bv} \right)^2 \frac{N}{\pi R^2} 2\pi b db \]

Integrate over all impacts to get net change per crossing:

\[ \Delta v^2 = \int_{b_{\text{min}}}^{R} d(v^2) \approx 8N \left( \frac{Gm}{Rv} \right)^2 \ln \left( \frac{R}{b_{\text{min}}} \right) \]

Time required to randomize initial velocity is relaxation time:

\[ t_r = \frac{v^2}{\Delta v^2} \frac{R}{\mathcal{V}} \sim \frac{N}{8 \ln N} t_c \]
Relaxation: Comments

1. 2-body relaxation is driven by shot noise.

2. Equal logarithmic intervals in impact parameter $b$ between $b_{\text{min}}$ and $\mathcal{R}$ contribute \textit{equally} to net relaxation.

3. Close encounters ($b \leq b_{\text{min}}$) occur on a timescale $\sim N t_c$. They play only a minor part in relaxation!

4. A broad mass spectrum speeds up relaxation; e.g., relaxation in MW's disk is driven by GMCs with $M \sim 10^6 M_\odot$.

5. Collective relaxation can occur, e.g., in marginally unstable systems, but doesn't preclude application of the Valsov equation.
Exponential Instability

Miller (1964); Goodman, Heggie, & Hut (1993)

Single encounter:

\[ \theta \approx \frac{Gm}{bv^2} \quad \delta \theta \approx \frac{Gm}{b^2v^2} \delta b \]

After time \( \tau \):

\[ \delta b' \approx \left( 1 \pm \frac{Gm\tau}{b^2v} \right) \delta b \]

Multiple encounters \( \Rightarrow \)

\[
\begin{align*}
\text{exponential growth} & \quad Gm\tau / b^2v \gg 1 \\
\text{random walk} & \quad Gm\tau / b^2v \ll 1
\end{align*}
\]

Encounters with \( Gm\tau / b^2v \approx 1 \) dominate: \( \tau \approx t_c, \ b \approx R N^{-1/2} \).

Integrating over all encounters, Liapounov time is \( t_e \approx 0.1t_c \).

Instability saturates on scale \( \sim R N^{-1/2} \).
Instability: Comments

1. 2-body relaxation and exponential instability occur on very different length and time scales; they are distinct effects.

2. In a galaxy, the exponential instability saturates on sub-parsec scales; once separations reach this scale, growth slows down.

3. If the smooth potential is nonintegrable, orbits continue to diverge exponentially up to scales $\mathcal{R}$, but on time $t_e \approx 10t_c$. 
Collisionless N-Body Method

Represent the distribution function with discrete bodies:

\[ f(\mathbf{r}, \mathbf{v}) \rightarrow \{ (m_i, \mathbf{r}_i, \mathbf{v}_i) \mid i = 1, \ldots, N \} \]

\[ f(\mathbf{r}, \mathbf{v}) = \sum_{i=1}^{N} m_i \delta^3(\mathbf{r} - \mathbf{r}_i) \delta^3(\mathbf{v} - \mathbf{v}_i) \]

This requires that for any phase-space volume \( V \),

\[ \int_V f(\mathbf{r}, \mathbf{v}) \, d^3\mathbf{r} d^3\mathbf{v} = \left\langle \sum_{(\mathbf{r}_i, \mathbf{v}_i) \in V} m_i \right\rangle \]

E.g., select \((\mathbf{r}_i, \mathbf{v}_i)\) with probability proportional to \( f(\mathbf{r}_i, \mathbf{v}_i) \), and assign all bodies equal mass: \( m_i = M / N \).

Many other sampling schemes are available (adaptive, quiet, etc.).
Equations of Motion

Move bodies along phase flow (method of characteristics):

\[
(\mathbf{r}_i', \mathbf{v}_i) = (\mathbf{v}_i, - \nabla \Phi|_{\mathbf{r}_i})
\]

Estimate potential from N-body representation:

\[
\nabla^2 \Phi|_{\mathbf{r}} \approx 4\pi G \sum_{i=1}^{N} m_i \delta^3(\mathbf{r} - \mathbf{r}_i)
\]

Singular potentials are numerically awkward, so substitute, e.g.:

\[
\delta^3(\mathbf{r} - \mathbf{r}_i) \rightarrow \frac{3}{4\pi} \frac{\epsilon^2}{(|\mathbf{r} - \mathbf{r}_i|^2 + \epsilon^2)^{5/2}}
\]

This yields equations of motion with “Plummer softening”:

\[
\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \quad \frac{d\mathbf{v}_i}{dt} = \sum_{j \neq i}^{N} \frac{G m_j (\mathbf{r}_j - \mathbf{r}_i)}{(|\mathbf{r}_j - \mathbf{r}_i|^2 + \epsilon^2)^{3/2}}
\]

Aarseth (1963)
Three Interpretations of Plummer Softening

**Modification of Normal Gravity**

\[ W_{12} = -\frac{G m_1 m_2}{\sqrt{r_{12}^2 + \epsilon^2}} \]

Hard to justify

Explicit Hamiltonian

Orbits are characteristics

Implicit in widely-used term “softened gravity”

**Interaction of Extended Bodies**

\[ \nabla^2 \Phi = 4\pi G (\rho \ast S_K) \]

Fully Newtonian

Explicit Hamiltonian

Not obvious

Aarseth (1963), Dyer & Ip (1993), Kalnajs (2013 MS)

**Smoothing During Force Calculation**

\[ \nabla^2 \Phi = 4\pi G (\rho \ast S_P) \]

Fully Newtonian

Not obvious

Orbits are characteristics

Hernquist & Barnes (1990), Barnes (2012)
**Effects of Softening**

1. **Softening biases the gravitational potential**

\[
\rho_p \propto (1 + r^2)^{-5/2}
\]

\[
\rho_H \propto r^{-1}(1 + r)^{-3}
\]

\[
\rho_J \propto r^{-2}(1 + r)^{-2}
\]

2. **Softening increases the relaxation time**

\[
\ln \mathcal{N} \simeq \ln(\mathcal{R}/b_{\text{min}}) \quad \rightarrow \quad \ln(\mathcal{R}/\epsilon)
\]

3. **Softening slows the exponential instability; numerical fit yields**

\[
t_e \approx (N\epsilon^2/\mathcal{R}^2)^{1/3} t_c
\]

*Goodman et al. (1993)*
Softening: Comments

1. Softening length must balance smoothness against bias:

\[ \epsilon \approx R N^{-1/2} \text{ to } R N^{-1/3} \]

2. Adaptive softening lengths (e.g., \( \epsilon \approx n^{-1/3} \)) are appealing, but may have unexpected and unintended consequences.

3. Alternate softening kernels, converging faster to \( 1/r \) potential:

Cubic spline kernel (Hernquist & Katz 1989)
Ferrers spheres (Dehnen 2001)
Bias-compensated

\[
W_{12} = -\frac{G m_1 m_2 (r_{12}^2 + \frac{3}{2}) \epsilon^2)}{(r_{12}^2 + \epsilon^2)^{3/2}}
\]

Binney & Tremaine (2008)

Claimed advantages not always evident in N-body practice…
**Time-Step Algorithms**

The underlying symmetry of the N-body equation of motion is evident in the Hamiltonian formulation:

\[
\frac{dr}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial r}
\]

This symmetry has consequences: (1) there are no attractors; (2) phase-space density is conserved; (3) evolution is reversible.

Symplectic integrators preserve this symmetry; e.g., leap-frog:

\[
\begin{align*}
    r_i^{[k+1]} &= r_i^{[k]} + \Delta t \, v_i^{[k + \frac{1}{2}]} \\
    v_i^{[k + \frac{3}{2}]} &= v_i^{[k + \frac{1}{2}]} + \Delta t \, a_i(r^{[k+1]})
\end{align*}
\]

Non-symplectic integrators are sometimes necessary but should be used with caution!
1. Head-on collision of Jaffee models
Experiments in Time Reversal

2. Reverse velocities and integrate
Experiments in Time Reversal
Conclusions & Comments

1. Vlasov model assumes that \( f(r, v, t) \) is complete description.

2. Shot noise on scales of \( R/N \) to \( R \) drives 2-body relaxation.

3. Strong chaos on small scales has limited effect on larger scales.

4. Sampling and softening are key concerns with N-body method.

5. Generation of good initial conditions is hard problem.

6. Trajectories converge as \( N \to \infty \), but very slowly!

7. N-body models can’t discriminate regular from chaotic orbits.
Thank You!