Astronomy 241: Review Problems #3

Review the questions below, and be prepared to discuss them in class on December 9 and 11. Modified versions of some questions will be used in the final exam.

Extra credit: turn in your solutions at the beginning of class on December 11. Correct solutions will boost your overall course grade by up to 5%.

1. An electron at rest is struck by an X-ray photon. The photon has energy $E_\gamma = 1$ keV $\simeq 1.60 \times 10^{16}$ kg m^2 s^{-2} and momentum $p_\gamma = E_\gamma/c$, where $c = 3 \times 10^8$ m s^{-1} is the speed of light. Use conservation of energy and momentum to show that the electron cannot simply absorb the photon. (Hint: you may safely assume the electron is non-relativistic, and use classical expressions for its kinetic energy and momentum.)

2. If protons were unable to "tunnel" through the Coulomb barrier, the Sun's internal temperature would have to reach $T_c \simeq 1.1 \times 10^{10}$ K before the p + p $\rightarrow$ 2H + e$^+$ + ν_e reaction could get going. Estimate the radius the Sun would contract to in order to reach this temperature.

3. Kepler's laws were deduced empirically from a detailed analysis of planetary positions. Subsequently, these laws were derived from Newton's laws of motion and gravity. In the process, Kepler's laws were generalized to include (i) bodies of comparable masses m_1, m_2, and (ii) unbound orbital motion.

(a) State Kepler's three laws for the motion of planets around the Sun in their original empirical forms.

(b) State the Newtonian generalizations of Kepler's three laws.

4. A satellite of mass m is on a circular orbit of radius r around a planet of mass $M \gg m$. Suppose we increase the satellite's speed by a factor of $\alpha > 1$ (i.e., $\vec{v}_{\rm final} = \alpha \vec{v}_{\rm initial}$).

(a) Compute the major axis, minor axis, pericenter distance, and apocenter distance of the new orbit.

(b) Note that $r_{\rm apo} \rightarrow \infty$ as $\alpha \rightarrow \sqrt{2}$. Explain this result by applying the virial theorem ($2K + U = 0$) to the initial orbit.

5. Callisto rotates slowly and has albedo $A \simeq 0.2$.

(a) What is Callisto's surface temperature at the subsolar point (that is, the point on Callisto directly facing the Sun)?

(b) At this temperature, could Callisto retain an atmosphere of He? Of N_2? Of Ar?

6. A number of planetary systems contain "hot Jupiters" — Jovian planets which orbit their parent stars at distances r < 0.1 AU.

(a) How close could Saturn orbit the Sun without being tidally disrupted?

(b) How close could Saturn orbit the Sun and retain its A ring (neglecting solar heating)?

(c) Allowing for the effects of solar heating, what would actually happen to Saturn's rings?

7. Describe the internal heating mechanisms in terrestrial planets, giant planets, and (some) moons of giant planets. How is energy transported inside these objects?

8. Consider a comet approaching the Sun on an orbit with $r_{\rm apo} = 15$ AU and $r_{\rm peri} = 0.5$ AU.

(a) Water ice sublimates rapidly at $T \geq 190$ K. If the comet's nucleus has an albedo of $A = 0.1$, how far from the Sun does the comet become active?

(b) At pericenter, solar heating is almost entirely balanced by sublimation of H_2O. If Q is the outgassing rate (units are molecules s^{-1}), the cooling term is $H_s Q/N_A$, where $H_s \simeq 5 \times 10^4$ kg m^2 s^{-2} $mole^{-1}$ is the latent heat of sublimation and $N_A = 6.02 \times 10^{23}$ is Avogadro's number. If the nucleus is ∼10 km in diameter, what is Q?

(c) The comet's ion tail is swept away from the Sun by the solar wind, which has a typical velocity of 400 $\mathrm{km\,s^{-1}}$. Observations show that the ion tail does not point exactly outward from the Sun; as a result of the comet's motion, it makes an angle of ϕ with respect to a line drawn outward from the Sun through the nucleus of the comet. Find ϕ.

9. Briefly describe the two main types of planets found in our solar system, and describe one of the two current explanations for this planetary dichotomy.

10. The Sun is powered by a sequence of nuclear reactions which convert hydrogen to helium.

(a) Write down the four steps of the PP I chain.

(b) Which step is the slowest, and why is it so slow?

(c) What is the minimum temperature required for the PP I chain?

(d) Why is the reaction rate so sensitive to temperature?

(e) How did the Sun get hot enough for this reaction to begin?

11. A typical model of the Sun is based on four key differential equations.

(a) Write down the left-hand sides of these equations, and briefly describe the four functions they define.

(b) Three additional equations must be specified to complete the solar model. What physical quantities do these relations specify?

(c) Two of the equations in part (a) also appear in models of giant planets such as Jupiter. Write down these equations in full.

12. Consider a highly simplified model of the greenhouse effect, in which the greenhouse gas forms a thin layer which is completely transparent to solar radiation, and completely opaque to the infrared radiation emitted by the planet's surface. Assume that the top and bottom surface areas of the layer are both equal to the planet's surface area.

(a) Let F be the flux of solar radiation, R_p be the planet's radius, and A the planet's albedo. What is the planet's heat input W_{abs} (units: kg m^2 s^{-3} = W) due to solar radiation?

(b) Assume that the greenhouse gas layer radiates like a blackbody with temperature T_g. What is T_g?

(c) Finally, show that the surface temperature of the planet is $T_p = 2^{1/4} T_g$.

13. Conservation of angular momentum can be used to test the idea that the Pluto-Charon system formed as a result of a collision between two "dwarf planet" progenitors. We know that Pluto and Charon are on circular orbits about their mutual center of mass, with an orbital period $P = 5.52 \times 10^5$ s and semi-major axis $a = 1.94 \times 10^7$ m. We also know that Pluto has mass $M_\mathrm{P} = 1.3 \times 10^{22}$ kg and radius $R_\mathrm{P} = 1.15 \times 10^6$ m, Charon has mass $M_\mathrm{C} = 1.4 \times 10^{21}$ kg and radius $R_\mathrm{C} = 6.0 \times 10^5$ m, and that both objects rotate once per orbital period.

(a) Calculate the orbital angular momentum L_P and L_C and rotational angular momentum S_P and S_C of Pluto and Charon. You may assume both objects are homogeneous spheres, with moments of inertia $I = \frac{2}{5}MR^2$. List these four angular momenta in order from largest to smallest, and sum them to get the total angular momentum J_PC of the Pluto-Charon system.

(b) Assume that the two progenitors which collided to form Pluto and Charon were identical, with masses $M = \frac{1}{2}(M_\mathrm{P} + M_\mathrm{C})$ and internal densities $\rho = 2000$ kg m^{-3}. Calculate the radii R of these objects.

(c) Now assume the two progenitors fell together on a parabolic orbit (total orbital energy $E = 0$), and just grazed each other at closest approach ($r_\mathrm{peri} = 2R$). Calculate the velocity v_peri of each object with respect to their mutual center of mass, and use the result to calculate the total angular momentum L_orb of their pre-encounter orbit.

(d) Compare the current angular momentum J_PC of the Pluto-Charon system to the orbital angular momentum L_orb of the progenitors. Does it seem plausible that the Pluto-Charon system could have formed in this way? (Note: tidal interactions insure that even a grazing encounter like the one considered here will result in the progenitors merging quite rapidly.)

14. The concepts of mean free path l and cross section σ can be extended to include not just simple physical collisions but other kinds of interactions between particles as well. In the center of the Sun, a proton has several different mean free path lengths. For this problem, assume the Sun's central density and temperature are $\rho_c \simeq 1.5\times10^5$ kg m^{-3} and $T_c \simeq 1.6\times10^7$ K, and assume the Sun is composed of pure hydrogen.

(a) The radius of a proton is $R_p \simeq$ 10-15 m. Two protons physically collide with each other when the distance between them is $r_\mathrm{geo} = 2R_p$. Assume that protons move on straight lines, (i.e., neglect electromagnetic forces); what are the cross section σ_geo and mean free path l_geo for physical (i.e., geometric) collisions?

(b) In the Sun, electromagnetic forces deflect protons long before they can approach to within r_geo. The electromagnetic interaction energy for a pair of protons is

$$U_\mathrm{em}(r) = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}, \tag{1}$$

where $e \simeq 1.60 \times 10^{-19}$ C is the elementary charge and $\epsilon_0 \simeq 8.9 \times 10^{-12}$ kg^{-1} m^{-3} s^2 C^2 is the permittivity of free space. Assuming protons have average energy $\frac{3}{2} k_B T_c$, find the distance $r_{\rm em}$ at which two protons are significantly deflected by electromagnetic forces.

(c) Using your result for part (b), how far does a typical proton in the Sun's core travel between significant deflections? This is the mean free path $l_{\rm em}$ for electromagnetic encounters between protons. What is the corresponding cross section $\sigma_{\rm em}$?

(d) Finally, a typical proton in the Sun's core survives for $t_{\rm nuc} \simeq 3 \times 10^{17}$ s before becoming involved in a nuclear reaction. Assuming the proton moves at the thermal velocity, how far does it travel in this time? This is the mean free path $l_{\rm nuc}$ for nuclear reactions between protons. What is the corresponding cross section $\sigma_{\rm nuc}$?

15. Inside the Sun, energy is transported outward by two key mechanisms, radiation and convection. Radiative transport implies that the temperature gradient is

$$\frac{dT}{dr} = -\frac{3}{16} \frac{\bar{\kappa}\rho}{\sigma_{\rm SB} T^3} \frac{L}{4\pi r^2} \tag{2}$$

where T is temperature, r is radius, κ is opacity, ρ is density, $\sigma_{\rm SB} = 5.67 \times 10^{-8}$ kg s^{-3} K^{-4} is the Stefan-Boltzmann constant, and L is the Sun's luminosity. Convective transport implies that the temperature gradient is

$$\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \frac{dP}{dr} \tag{3}$$

where $\gamma = 5$ is the ratio of specific heats, and P is pressure.

(a) At radius $R_1 = 0.714 R_\odot$, where the Sun's radiative zone ends and its convective zone begins, these two expressions yield (almost) identical values for the temperature gradient, dT/dr. Why?

(b) The total mass within radius R_1 is $M_1 = 0.976 M_\odot$. (That's right: nearly 98% of the Sun's mass lies within the radiative zone!) Evaluate the gravitational acceleration g_1 at this radius.

(c) At radius R_1, the temperature is $T_1 = 2.18 \times 10^6$ K and the mean molecular weight is $\mu = 0.60$. Evaluate the pressure scale height

$$H_1 = \frac{k_B T_1}{\mu m_H g_1} \tag{4}$$

where $k_B = 1.38 \times 10^{-23}$ kg m^2 s^{-2} K^2 is Boltzmann's constant. How does the scale H_1 compare to the Sun's radius?

(d) At radii near R_1, the pressure is

$$P(r) \simeq P_1 e^{(R_1 - r)/H_1} \tag{5}$$

where $P_1 = 5.59 \times 10^{12}$ kg m^{-1} s^{-2} is the pressure at R_1. Use Eq. (3) above to evaluate the convective temperature gradient, $(dT/dr)_{\rm con}$.

(e) At radius R_1, the density is $\rho = 187$ kg m^{-3} and the opacity is $\bar{\kappa} = 2.0$ kg^{-1} m^2. Use Eq. (2) above to evaluate the radiative temperature gradient, $(dT/dr)_{\rm rad}$.