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Two vortex rings produce chaos

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Abstract. – We study the streamlines of the velocity field produced by two unlinked vortex rings. We find that two vortex rings can produce chaos and observe a route to chaos directly from periodic orbits. From the case study of numerous ring configurations we give conditions for integrability of streamlines. We also find that near the larger ring or the stronger ring there are more stable regular structures.

Vortices are abundant in many fluid flows in nature. The most spectacular examples are tornadoes. Visualization of coherent vortical structures in turbulent fluid flows leads to the hope that complex turbulent flows might be modeled by vortex systems. Even relatively simple and static or slowly moving vortex structures could be capable of producing highly chaotic flow fields. There has been a significant amount of work on point vortices in two dimensions [1], but relatively little work on three-dimensional vortex flow. The frequently studied problem of coaxial vortex rings can also be reduced to a two-dimensional description. In the early 1990s, there have been studies on the superposition of spherical vortices in a steady flow and chaotic streamlines have been found in some cases [2–5]. A fundamental question is what vortex structures possess chaotic flow fields, in contrast to a periodic or quasi-periodic behavior [6].

In this letter, we consider the three-dimensional steady-state motion induced by two unlinked vortex rings. For a review on vortex rings, see ref. [7]. Our major objective is to investigate the circumstances that give rise to regular flow fields. This knowledge would be useful for illuminating and anticipating the flow properties of more general vortex structures. We use the well-known Helmholtz vortex ring solution, which is a circular vortex ring with infinitely small core. Its velocity field can be expressed in terms of a stream function ψ [8],

$$\psi = \frac{\kappa}{2\pi}\sqrt{ar}\left[\frac{2-k^2}{k}K(k) - \frac{2}{k}E(k)\right], \tag{1}$$

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with $k^2 = 4ar/[z^2 + (r+a)^2]$. Here, r is the distance from the ring axis and z the distance from the plane of the ring. The parameters κ and a are the strength and radius of the vortex ring, respectively. $K(k)$ and $E(k)$ are the complete elliptic integrals of first and second kind. The three velocity components are

$$u_i = \frac{1}{r}\frac{\partial\psi}{\partial r} l_i - \frac{1}{r^2}\frac{\partial\psi}{\partial z}(\delta_{ij} - l_i l_j)x_j\,, \qquad i = 1,2,3\,, \tag{2}$$

where l_i is the unit vector of the rectilinear axis direction, that is, the direction of the velocity at the centre of the vortex ring.

Here we study two unlinked vortex rings, whose velocities are u'_i and u''_i, respectively. The solution of the velocity field can be superposed. So the dynamic equations which we study are as follows:

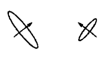
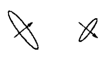
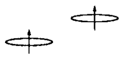
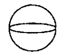
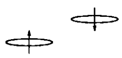
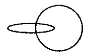
$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = u'_i + u''_i\,, \qquad i = 1,2,3\,. \tag{3}$$

We choose the line connecting both centres s' and s'' as the x_2-axis and the positive direction of x_2 is from s' to s''. We take the middle point of $s's''$ as the origin o of the coordinate system with $s'o = os'' = d$. If d is not too small, the influence on one ring by the other ring is insignificant (see ref. [8], pp. 236-243). So in this letter we ignore the deformation of the vortex rings due to the velocity field of the other ring and consider the rings to be fixed in space.

There is a variety of possible configurations. We can vary the location of the two centres s' and s'', the rings' directions l'_i and l''_i, the strengths κ' and κ'', and the radii a' and a''. Regarding axis orientations, we distinguish two situations. One situation is that l'_i and l''_i lie on the same plane, which is chosen as the $x_1 = 0$ plane. The other situation is that l'_i, l''_i and the x_2-axis do not lie on the same plane. The latter case certainly can produce chaos as discussed elsewhere [9]. So now we only consider the former case. Here we should keep in mind that owing to incompressibility, our system of two vortex rings is a conservative dynamical system. Table I provides an overview of the results for the various ring arrangements investigated.

In order for the trajectories to be integrable, special symmetries must exist. Suppose there

TABLE I – *Overview of ring arrangements along with their streamline topology. The third column shows the number of symmetry planes* (p)*, the existence of a special symmetry plane* (s)*, and the existence of an invariant surface separating the rings* (w)*. The last two cases* [9] *illustrate, respectively, vortex rings with common centre and linked rings with ring axes not on the same plane.*

Configuration		Streaml.	p/s/w	Configur.	Streaml.	p/s/w
	coplanar, Ia	periodic	3/yes/yes		periodic	2/yes/yes
	coplanar, Ib	periodic	3/yes/yes		periodic	2/yes/?
	mirrored, IIa	quasi-per.	2/no/yes		irregular	1/no/yes
	mirrored, IIb	periodic	2/yes/yes		irregular	1/no/?
	parallel, IIIa	irregular	1/no/no		periodic	3/yes/yes
	parallel, IIIb	irregular	1/no/no		irregular	0/no/no

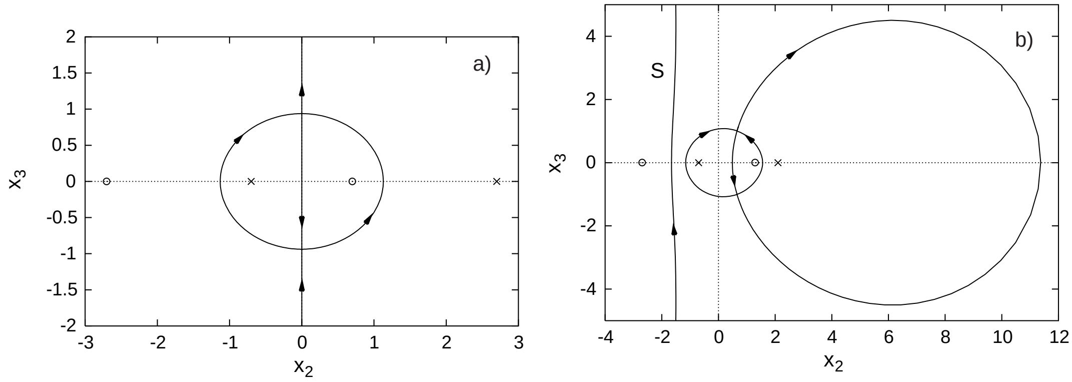


Fig. 1 – Stable and unstable invariant manifolds (solid lines) on the $x_1 = 0$ plane for coplanar vortex rings with equal orientation. Intersections of the vortex rings with the plane are marked with $\odot$ and $\times$ corresponding to their orientation. In part (a) the two rings have equal radius (case Ia) and in part (b) they have different radii. The separatrix labeled "S" extends to infinity.

is a symmetry plane of the form

$$\begin{aligned} u_y(-y,v,w) &= +u_y(y,v,w)\,, \\ u_v(-y,v,w) &= -u_v(y,v,w)\,, \\ u_w(-y,v,w) &= -u_w(y,v,w). \end{aligned} \tag{4}$$

The coordinate system (y, v, w) does not need to coincide with (x_1, x_2, x_3). We shall show that any trajectory passing through such a symmetry plane is regular. The velocities are perpendicular to this symmetry plane. As a consequence of eq. (4), the streamlines will also be symmetric with respect to this plane. It directly follows that any streamline which intersects the plane more than once must be periodic and passes through the plane exactly twice! Even when a trajectory intersects the symmetry plane only once, it cannot be chaotic. One can see this in the following way. The eigenvalues at a particular point in space are the negative of the eigenvalues at the mirror point, since expansion on one side of the symmetry plane corresponds to contraction on the other side. The Lyapunov exponents are the time averages of the eigenvalues. Averaging over an entire streamline intersecting the symmetry plane once will explore each half equally. As a result all Lyapunov exponents are zero, and the trajectory is thus regular. Our argument excludes trajectories that never intersect the symmetry plane; these trajectories can, therefore, be chaotic.

We start our exploration from the simplest case: the two vortex rings lie on the same plane, the $x_3 = 0$ plane, and have equal radius (case I). Numerical calculations show that all of the trajectories of the fluid particle are periodic orbits. The plane $x_3 = 0$ is a symmetry plane of type (4). With both rings having the same orientation (case Ia), *i.e.* same sign of κ' and κ'', all of the trajectories wind around either one of the two rings but not both. There are two fixed points on the $x_1 = 0$ plane (see fig. 1a). With opposite orientation (case Ib, not shown) some of them wind around one ring, while others wind around both rings. The x_1-axis consists of fixed points. The fluid motion in both cases is regular.

Next, we let the two vortex rings be inclined at some angle towards each other maintaining mirror symmetry (case II). The fluid motion for the same and for opposite orientations (cases IIa and IIb) is still regular. A symmetry plane of type (4) is present in case IIb (the $x_2 = 0$ plane), but not in case IIa. In case IIa trajectories are quasi-periodic, while they are periodic in case IIb as required for trajectories that repeatedly intersect the symmetry plane. The trajec-

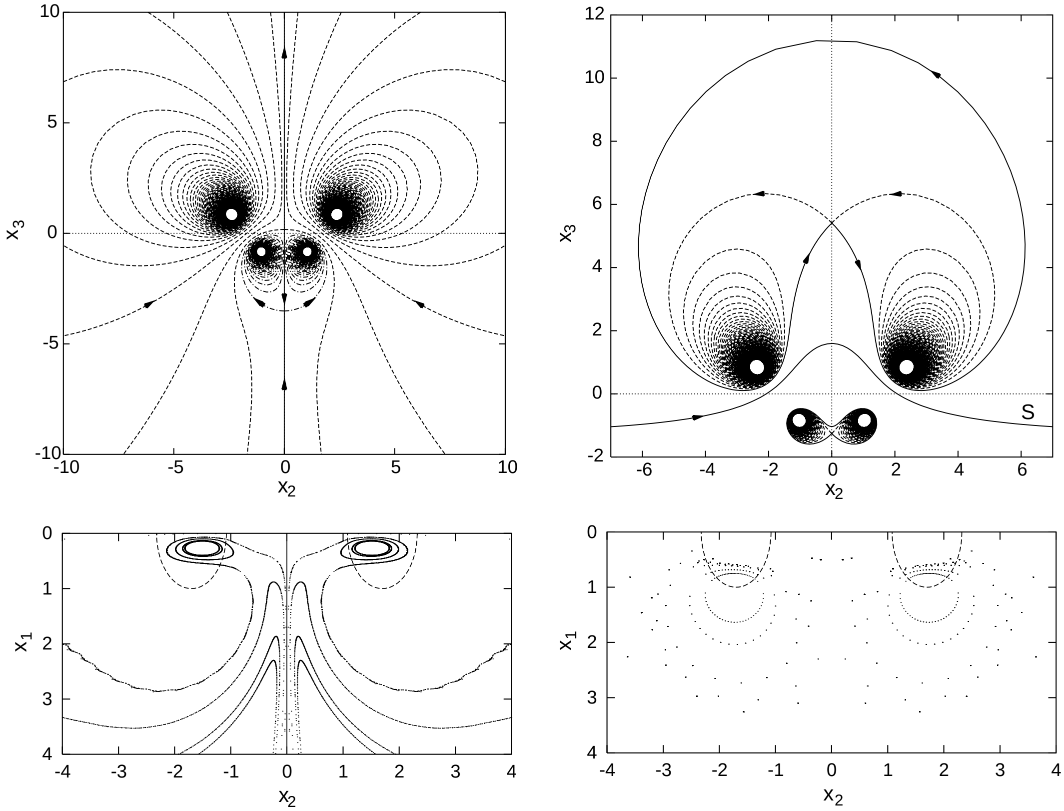


Fig. 2 – Stable and unstable invariant manifolds on the $x_1 = 0$ plane (upper two figures) and Poincaré sections on the $x_3 = 0$ plane (lower two figures). The two vortex rings are mirror images of each other. The left panel (case IIa) and the right panel (case IIb) differ by the relative orientation of the rings. The four spots where the vortex rings intersect the plane are left white because streamlines are integrated over a finite time only. The dashed lines in the Poincaré section are projections of the vortex rings.

tories in these two cases are closed streamlines and stream tori. Figure 2 shows the manifolds of the fixed points on the $x_1 = 0$ plane. There are two saddle points and four foci (two stable and two unstable). In case IIb, there still exists a closed line consisting of fixed points.

Now, we change the relative position of the two vortex rings. We let the two vortex rings be inclined at the same angle to the x_3-axis (case III). This case is just like moving one ring a distance along its rectilinear axis. No symmetry plane of type (4) exists and the fluid motion is found to be chaotic. Figure 3 shows the manifolds of the fixed points on the $x_1 = 0$ plane. Obviously, the structure of these manifolds is more complicated than in cases IIa and IIb.

If the inclination angle is 90°, both centres of the vortex rings are located on the same rectilinear axis. The flow is axisymmetric. All of the streamlines lie on the meridian plane and they are periodic orbits. The fluid motion is regular. (If the vortex rings are allowed to move, chaotic mixing occurs [10].)

In our calculations, we have observed that changing the strength of the ring has the same effect as changing its radius. The stronger ring corresponds to the larger ring. Here we only discuss the cases of different radii. For coplanar rings (case I) changing the radius preserves the symmetry plane (4) and the streamlines indeed remain periodic. If we start from case

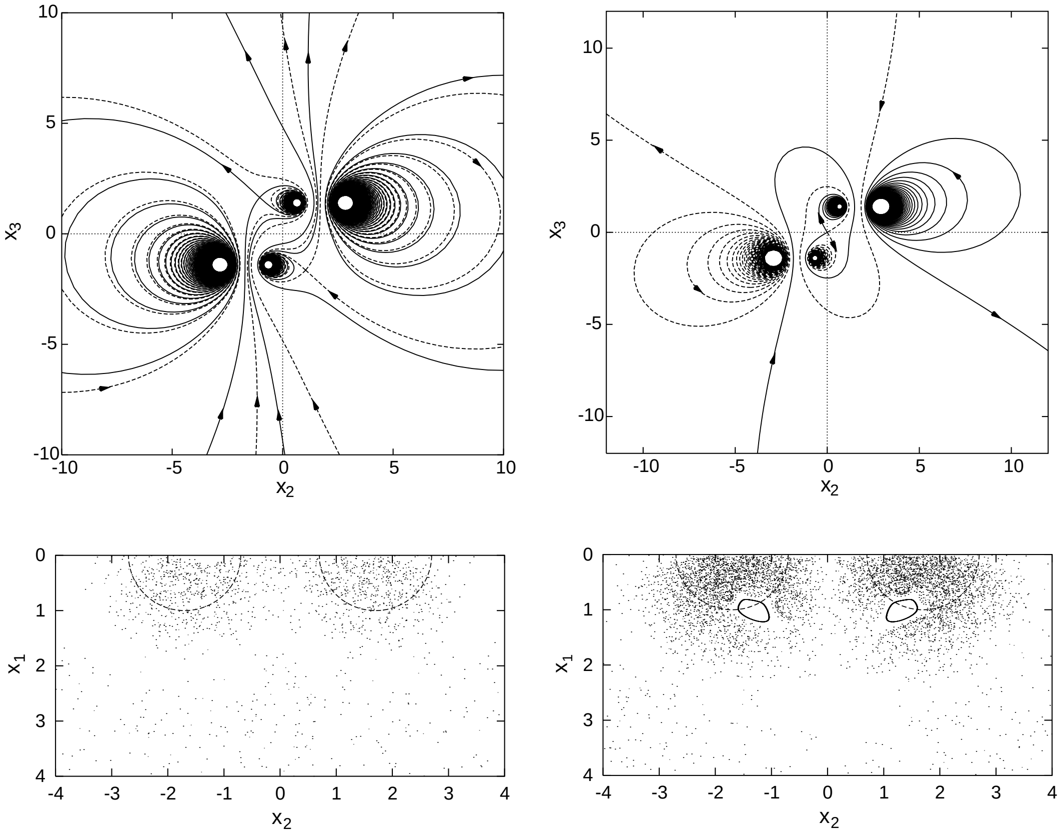


Fig. 3 – Stable and unstable invariant manifolds on the $x_1 = 0$ plane (upper two figures) and Poincaré section (lower two figures). Separatrices reaching to infinity are not drawn. Left panel: The axes of the two vortex rings are parallel to each other (case IIIa). Each of the two hyperbolic fixed points is connected with all four vortex ring intersections. Right panel: Vortex rings with opposite orientation (case IIIb). The hyperbolic fixed point is connected with all four vortex ring intersections.

IIa and change the radius of one ring, chaos appears (fig. 4). Changing strength or radius destroys the symmetry of the flow field. The calculations show that near the larger ring the streamlines are regular. It means that the larger ring has stronger stabilizing effect on the flow field. The same is true when the two rings have opposite orientations.

In all the integrable cases discussed, there exists an invariant surface which separates the two vortex rings. This means that the situation could equally well be considered as one ring with a wall. (The other ring does however not need to be the "image" of the original ring.) The boundary condition is that the velocity is either everywhere tangential or everywhere perpendicular to the wall. For instance, in case Ia (fig. 1a) the $x_2 = 0$ plane plays the role of a wall. If the radius of one ring is changed (fig. 1b), then an invariant surface with the topology of a sphere wraps around the smaller ring. A cross-section of this sphere is visible in fig. 1b as the larger of two ovals. (The smaller oval also originates from a spherical invariant surface, but this surface intersects the rings.) For the other ring configurations, the presence of a wall —may it be planar or curved— is indicated in table I. Especially interesting is case IIa with different radii. In fig. 4 the closed orbit labeled "C" and the separatrix "S" are both part of a surface which reaches to infinity and separates the two rings from each other. All streamlines

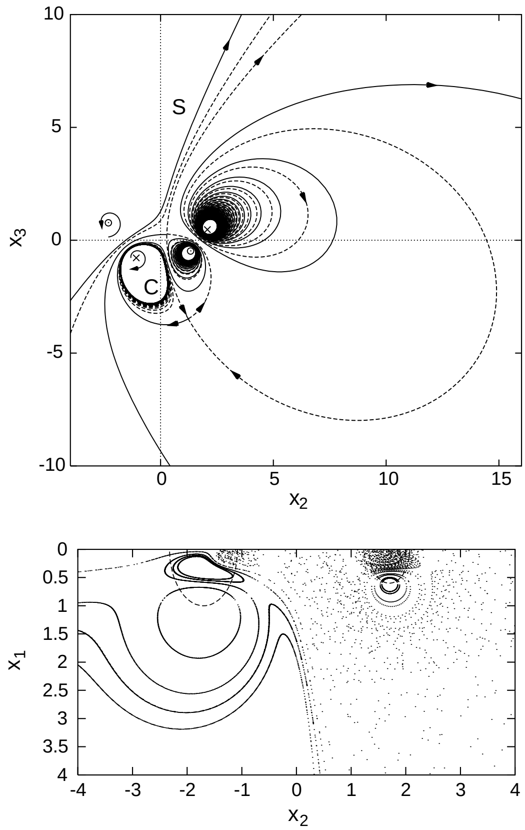


Fig. 4 – Same as fig. 2 (left panel), but the ring on the right has a smaller radius. The label "S" refers to a separatrix reaching to infinity. "C" is a closed orbit, which lies on the same two-dimensional invariant manifold as "S".

around the larger ring are regular while all streamlines around the smaller ring are chaotic. The situation would not change when the separating surface is replaced by a wall. Hence, we have here an example of a single vortex ring which has chaotic streamlines! The volume in which it is immersed is not simply connected, because streamlines can pass between "S" and "C". On the other hand, the streamlines of the larger ring fill a simply connected volume and are regular.

We observe that whenever there exist two or more symmetry planes, which are perpendicular to each other, no chaos appears. The symmetry considered here is of the velocity field with respect to a plane, ignoring the orientation of the velocity vector. Cases Ia and Ib have three symmetry planes, which are perpendicular to each other. They are the $x_1 = 0$ plane, $x_2 = 0$ plane, and $x_3 = 0$ plane. Cases IIa and IIb have two symmetry planes, which are the $x_1 = 0$ plane and $x_3 = 0$ plane. All of the above four cases show regular motion. Cases IIIa and IIIb have only one symmetry plane, the $x_1 = 0$ plane. There appears chaos. The cases of different strengths and radii follow the same rule. Cases IIa and IIb with different radii have two symmetry planes and are regular. Cases IIIa and IIIb have only one symmetry plane and are chaotic. From the case studies in this letter and in ref. [9] we find that the most important factor to produce chaos is the absence of symmetry planes. According to our calculations and analyses the process forming chaos is thus a symmetry-breaking process.

Streamlines are regular for the following fixed arrangements of two vortex rings: coplanar rings (with any radii and strengths), coaxial rings (with any radii and strengths), rings which are mirror images of each other (equal strength and radius, but either combination of orientations), and rings of equal radius and strength that have the same centre but are at an angle to each other. Since we have systematically explored cases with simple spatial symmetries, this list comprises probably all integrable cases. The integrability of all but one case is explained by the existence of a symmetry plane of form (4). Only mirrored rings of the same (mirrored) orientation have no such symmetry plane, and this is also the only regular case with quasi-periodic trajectories. All the other integrable cases have only periodic trajectories. Our numerical calculations show that deviations from any of these configurations lead to irregular streamlines and there are no further "hidden symmetries".

Moving two coplanar rings (case I) onto distinct parallel planes (case III) changes periodic orbits (with a single frequency) to chaotic orbits. This is an example of a direct route to chaos from periodic orbits, which is a global bifurcation, *i.e.* homoclinic or heteroclinic bifurcation [11]. The topology of the streamlines has changed, although the topological arrangement of the rings remained unlinked.

In summary, we have observed and discussed different rules for the integrability of streamlines: i) A symmetry plane of type (4), which accounts for all but one case; it is a rigorous criterion as long as all streamlines intersect the plane. ii) The existence of two or more perpendicular symmetry planes, which identifies integrability uniquely; it is an empirical rule. iii) The existence of a boundary surface (a "wall") on the border of a simply connected volume containing a single ring.

From these results we can see clearly that for two vortex rings, even when they are far away from each other, chaos appears already when they have different axis directions. So we predict that deformed vortex rings, that cannot lie on a plane, will produce chaos.

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